

Arc-Length Method for Frictional Contact with a Criterion of Maximum Dissipation of Energy

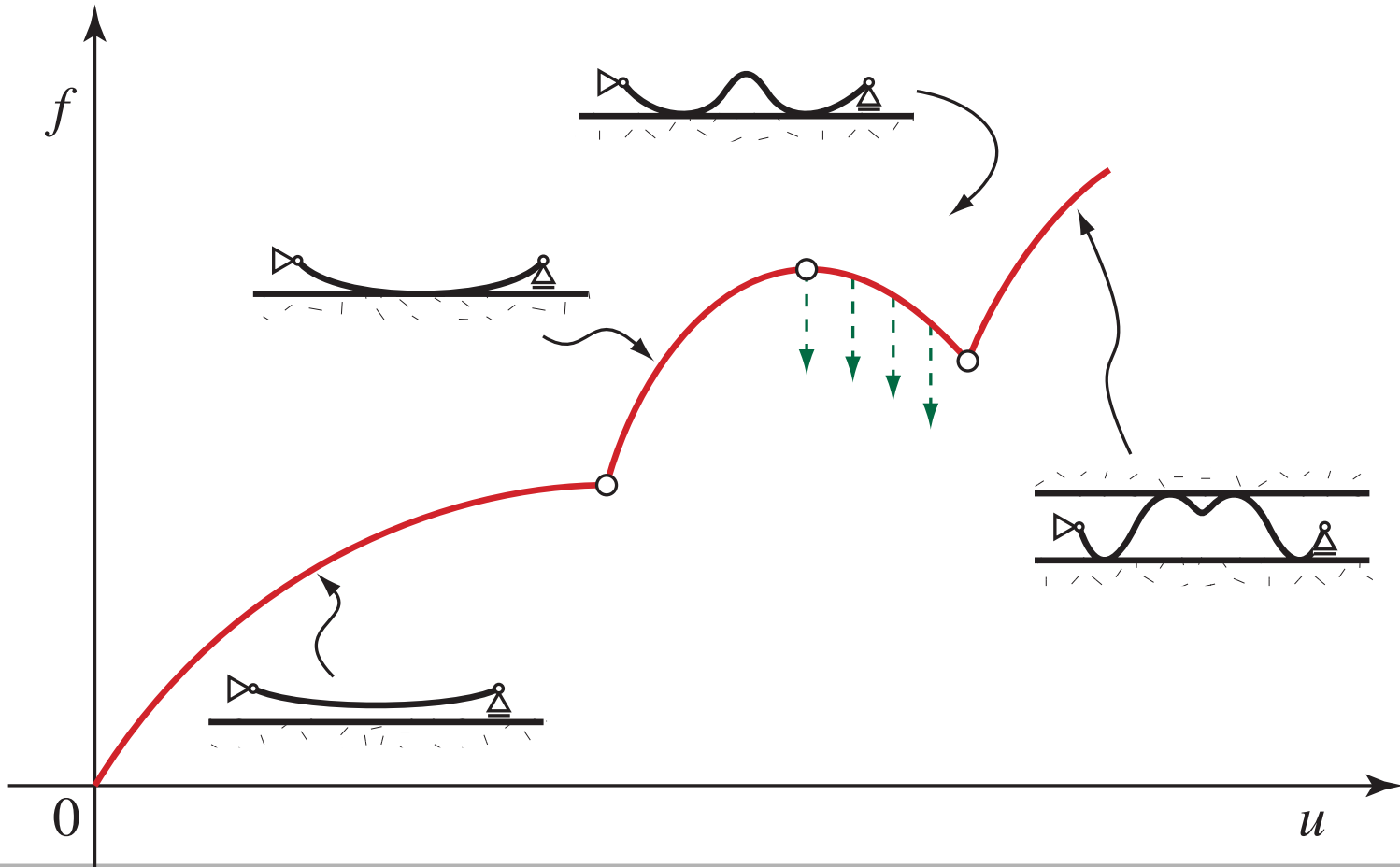
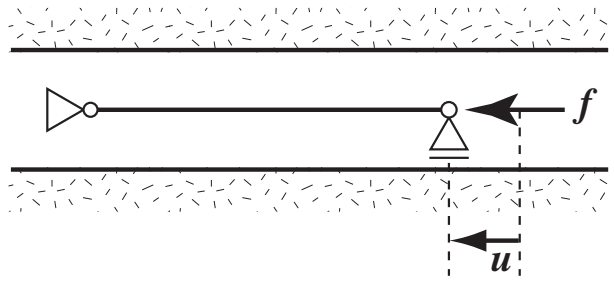
Yoshihiro Kanno[†] João A.C. Martins[‡]

[†]The University of Tokyo (Japan)

[‡]Instituto Superior Técnico (Portugal)

Frictional contact problems

- contact / free
- slip / stick
- \implies limit / bifurcation points



Existence results of contact problems

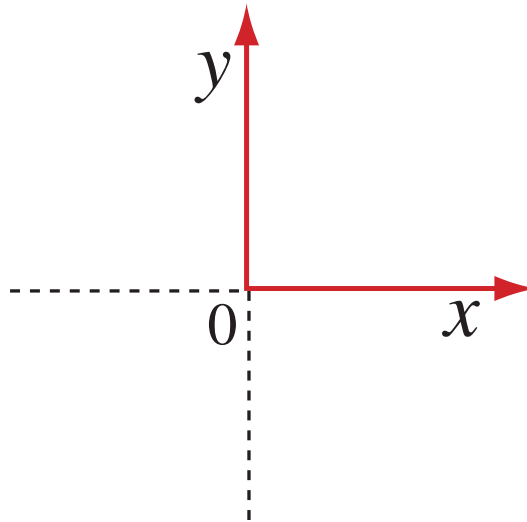
- frictionless
 - continuation method [Miersemann & Mittelmann 89]
- frictional & large disp.
 - tangent stiffness [Wriggers 02]
 - arc-length method [Koo & Kwak 96]
 - approximation of friction law [Perić & Owen 92]
- selection of paths (small deformation)
 - min. of potential energy [Hilding 00]

Existence results of contact problems

- frictionless
 - continuation method [Miersemann & Mittelmann 89]
- frictional & large disp.
 - tangent stiffness [Wriggers 02]
 - arc-length method [Koo & Kwak 96]
 - approximation of friction law [Perić & Owen 92]
- selection of paths (small deformation)
 - min. of potential energy [Hilding 00]
- \implies arc-length method + path selection
- \implies solve an optimization problem at each increment

Complementarity condition

$$\mathbf{x} \geq \mathbf{0}, \quad \mathbf{y} \geq \mathbf{0}, \quad \mathbf{x}^\top \mathbf{y} = 0$$



Generally & for short, we often write

$$\mathbf{0} \leq (\mathbf{Ax}) \perp (\mathbf{By}) \geq \mathbf{0}$$

- Mathematical Program with Complementarity Constraints = MPEC

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \\ & \mathbf{0} \leq \mathbf{h}(\mathbf{x}) \perp \mathbf{w}(\mathbf{x}) \geq \mathbf{0} \end{aligned}$$

complementarity condition:

$$\mathbf{0} \leq \mathbf{h}(\mathbf{x}) \perp \mathbf{w}(\mathbf{x}) \geq \mathbf{0}$$



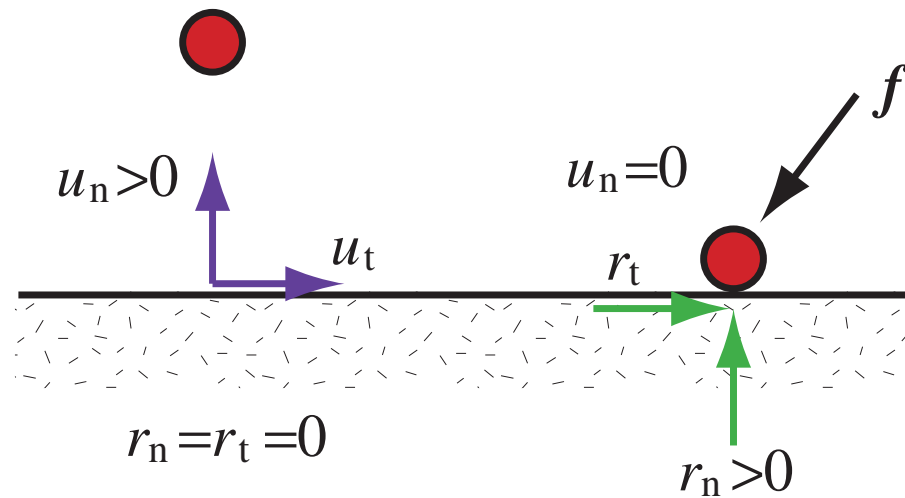
$$\mathbf{h}(\mathbf{x}) \geq \mathbf{0}, \quad \mathbf{w}(\mathbf{x}) \geq \mathbf{0}, \quad \mathbf{h}(\mathbf{x})^\top \mathbf{w}(\mathbf{x}) = 0$$

Contact problems (2D)

non-penetration condition :

$$u_n > 0 \implies r_n = 0 \quad : \text{ free}$$

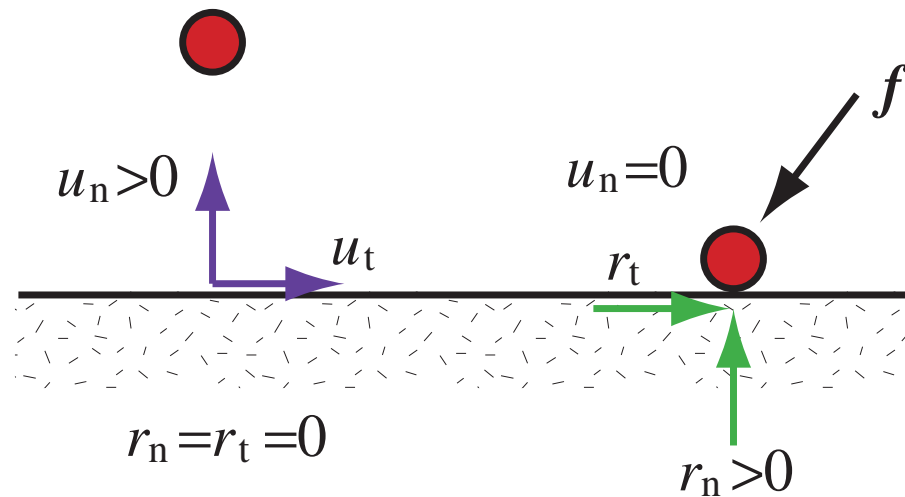
$$r_n > 0 \implies u_n = 0 \quad : \text{ contact}$$



Contact problems (2D)

non-penetration condition :

$$\begin{aligned} u_n > 0 &\implies r_n = 0 && : \text{ free} \\ r_n > 0 &\implies u_n = 0 && : \text{ contact} \end{aligned}$$



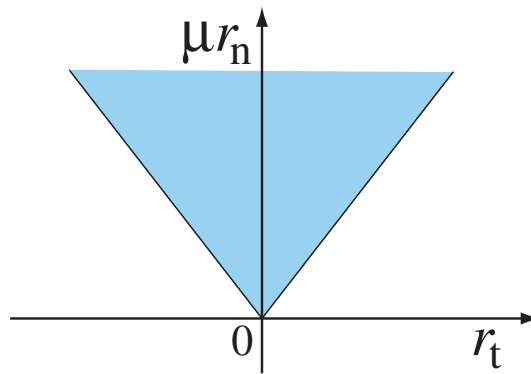
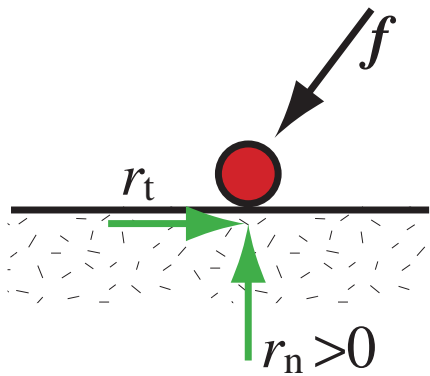
complementarity condition :

$$u_n \geq 0, \quad r_n \geq 0, \quad u_n r_n = 0.$$

Coulomb's friction law

$$\mu r_n \geq |r_t|$$

$$\Delta u_t = -\alpha r_t, \quad \begin{cases} \alpha > 0 & \text{(slip)} \\ \alpha = 0 & \text{(stick)} \end{cases}$$



μ : coefficient of friction

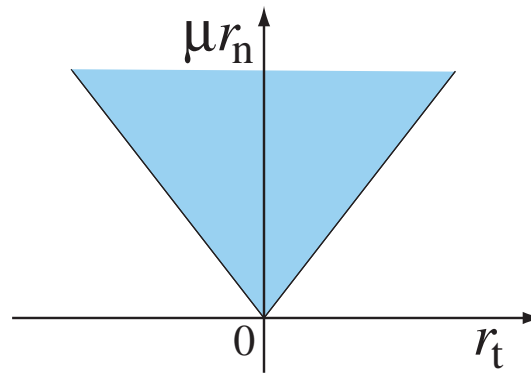
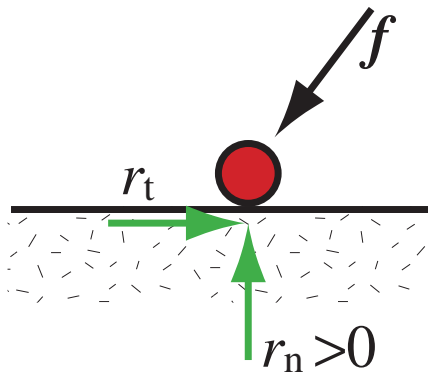
(r_n, r_t) : reaction (normal, tangential)

Coulomb's friction law

complementarity condition:

$$\mu r_n \geq |r_t|, \quad \lambda_n \geq |\Delta u_t|$$

$$(\mu r_n, r_t) \cdot (\lambda_n, \Delta u_t) = 0$$



μ

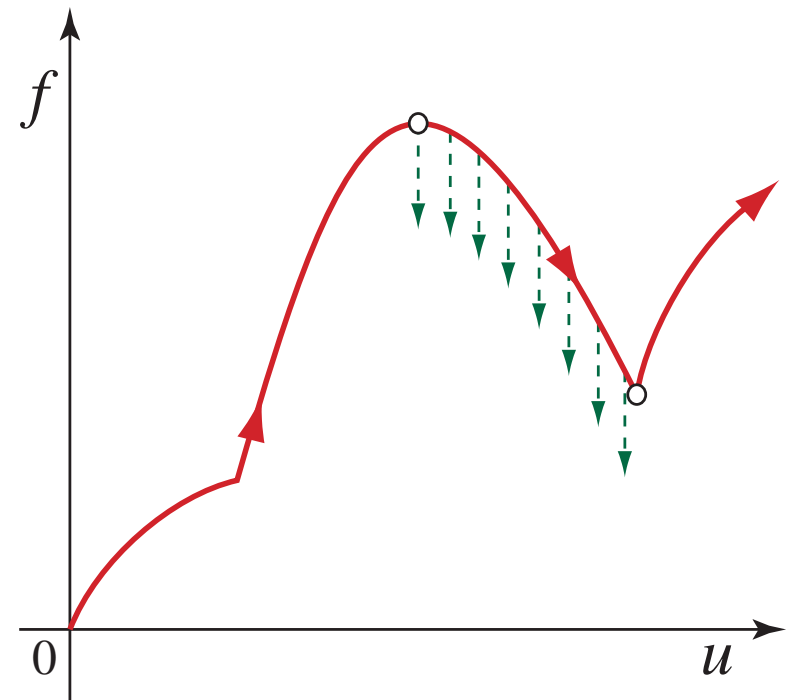
: coefficient of friction

(r_n, r_t)

: reaction (normal, tangential)

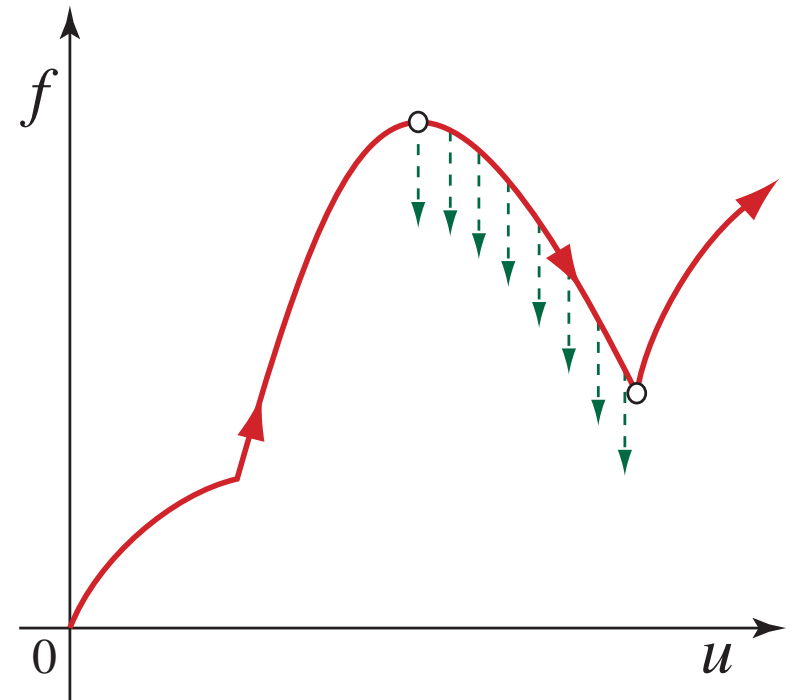
Aims of the talk

- quasi-static contact, Coulomb's friction
- limit points (smooth / nonsmooth)
- avoid bifurcated unloading paths



Aims of the talk

- quasi-static contact, Coulomb's friction
 - \implies complementarity conditions
- limit points (smooth / nonsmooth)
 - \implies arc-length method
- avoid bifurcated unloading paths
 - \implies path with maximum dissipation of energy



Arc-length method

At each increment,

- conventional arc-length method
 - u : displacements
 - s : loading parameter
 - find $(\Delta \mathbf{u}^{(k)}, \Delta s^{(k)})$
satisfying $\|(\Delta \mathbf{u}^{(k)}, \Delta s^{(k)})\| = \bar{S}$
by solving nonlinear equations
- our method
 - find $(\Delta \mathbf{u}^{(k)}, \Delta s^{(k)})$ and $\mathbf{r}^{(k)}$
satisfying $\|(\Delta \mathbf{u}^{(k)}, \Delta s^{(k)})\| = \bar{S}$
by solving an **MPEC**

Choose the path with max. dissipation

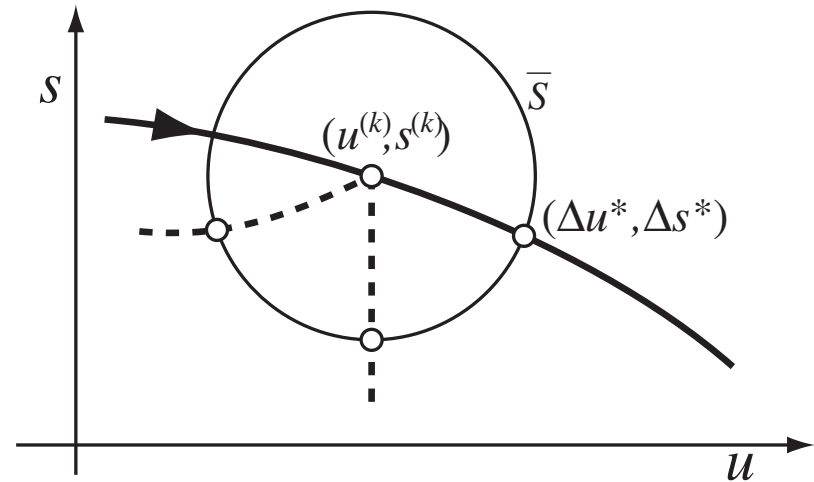
system of equilibrium paths (\clubsuit):

$$\phi(\Delta \mathbf{u}, \Delta s) - \mathbf{r} = \mathbf{0} \quad (\text{equilibrium eqs.})$$

$$\mathbf{0} \leq \mathbf{A}\Delta \mathbf{u} \perp \mathbf{B}\mathbf{r} \geq \mathbf{0} \quad (\text{non-penetration \& friction})$$

hypersphere constraint on arc-length (\diamond):

$$\|(\Delta \mathbf{u}, \Delta s)\| = \bar{S}$$



Choose the path with max. dissipation

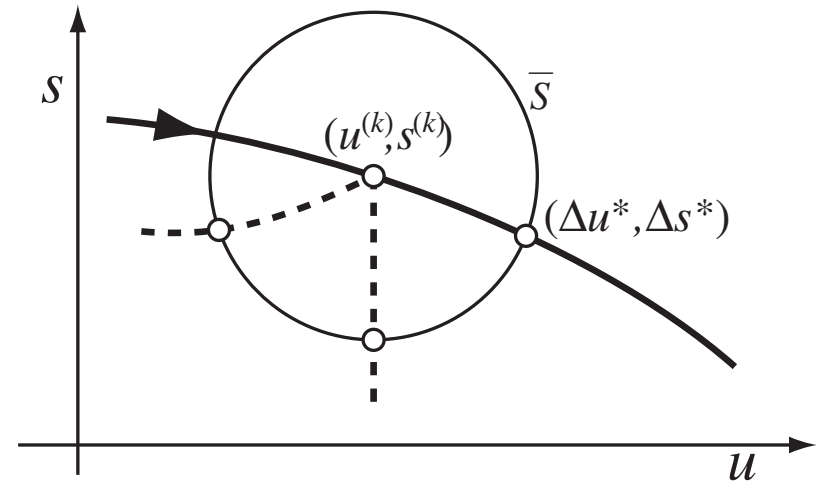
system of equilibrium paths (\clubsuit):

$$\phi(\Delta \mathbf{u}, \Delta s) - \mathbf{r} = \mathbf{0} \quad (\text{equilibrium eqs.})$$

$$\mathbf{0} \leq \mathbf{A}\Delta \mathbf{u} \perp \mathbf{B}\mathbf{r} \geq \mathbf{0} \quad (\text{non-penetration \& friction})$$

hypersphere constraint on arc-length (\diamond):

$$\|(\Delta \mathbf{u}, \Delta s)\| = \bar{S}$$



prototype of incremental problem:

$$\max \quad -\mathbf{r}_t^{(k)} \cdot \Delta \mathbf{u}_t \quad (\text{dissipation of energy})$$

$$\text{s.t.} \quad (\clubsuit) \ \& \ (\diamond)$$

$$\begin{aligned} \max \quad & (\text{dissipation}) - p(\Delta \mathbf{u}, \Delta s) \\ \text{s.t.} \quad & (\text{equilibrium eqs.}), \\ & (\text{complementarity conds.}) \end{aligned}$$

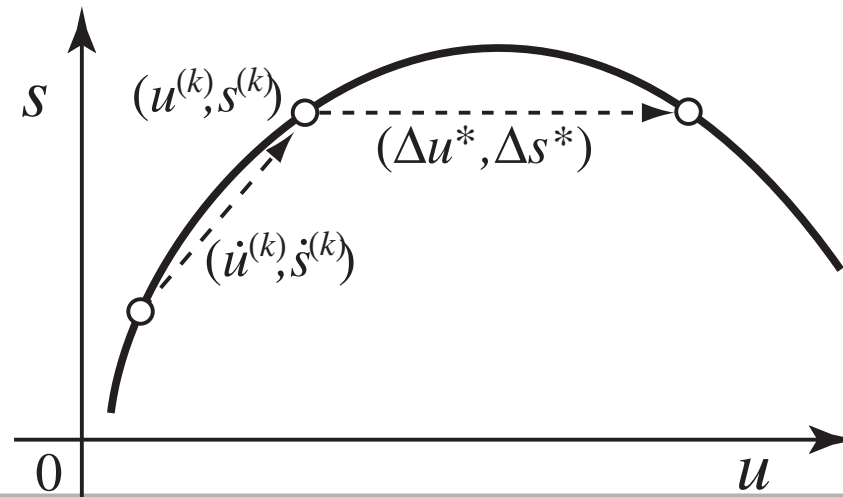
- penalty

$$p = \frac{1}{2} \gamma_1^k \|(\Delta \mathbf{u}, \Delta s)\|^2$$

⇐ on arc-length

$$- \gamma_2^k (\dot{\mathbf{u}}^{(k)}, \dot{s}^{(k)}) \cdot (\Delta \mathbf{u}, \Delta s)$$

⇐ prevents 'backward' solution



$$\begin{aligned} \max \quad & \text{(dissipation)} - p(\Delta \mathbf{u}, \Delta s) \\ \text{s.t.} \quad & \text{(equilibrium eqs.),} \\ & \text{(complementarity conds.)} \end{aligned}$$

- **penalty**

$$p = \frac{1}{2} \gamma_1^k \|(\Delta \mathbf{u}, \Delta s)\|^2 \quad \Leftarrow \text{on arc-length}$$

$$- \gamma_2^k (\dot{\mathbf{u}}^{(k)}, \dot{s}^{(k)}) \cdot (\Delta \mathbf{u}, \Delta s) \quad \Leftarrow \text{prevents 'backward' solution}$$

- **MPEC has**

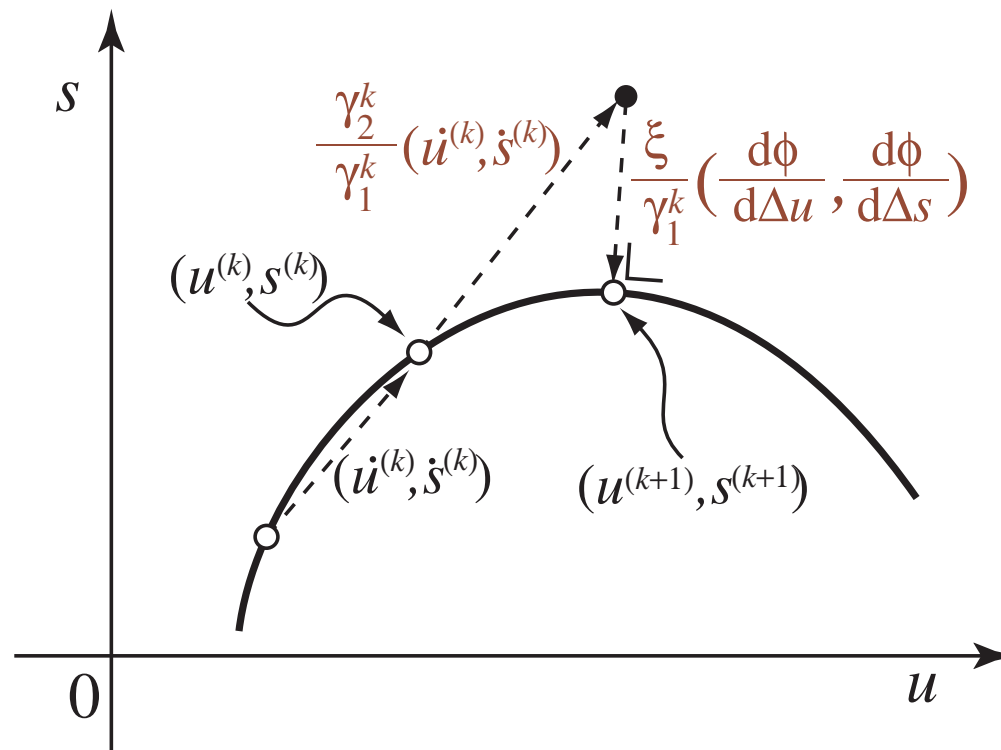
- convex objective function
- feasible solutions

- control $\|(\Delta \mathbf{u}, \Delta s)\|$ by choosing γ_1^k

Characterization of solution ($r_t^{(k)} = 0$)

KKT conditions:

$$\begin{pmatrix} \mathbf{u}^{(k+1)} \\ s^{(k+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{u}^{(k)} \\ s^{(k)} \end{pmatrix} + \frac{\gamma_2^k}{\gamma_1^k} \begin{pmatrix} \dot{\mathbf{u}}^{(k)} \\ \dot{s}^{(k)} \end{pmatrix} + \frac{1}{\gamma_1^k} \sum_{l=1}^{n^d} \xi_l \begin{pmatrix} \partial \phi_l / \partial \Delta \mathbf{u} \\ \partial \phi_l / \partial \Delta s \end{pmatrix}$$



Regularization of MPEC

regularization:

$$(\mu r_n, r_t) \cdot (\lambda_n, \Delta u_t) = 0$$

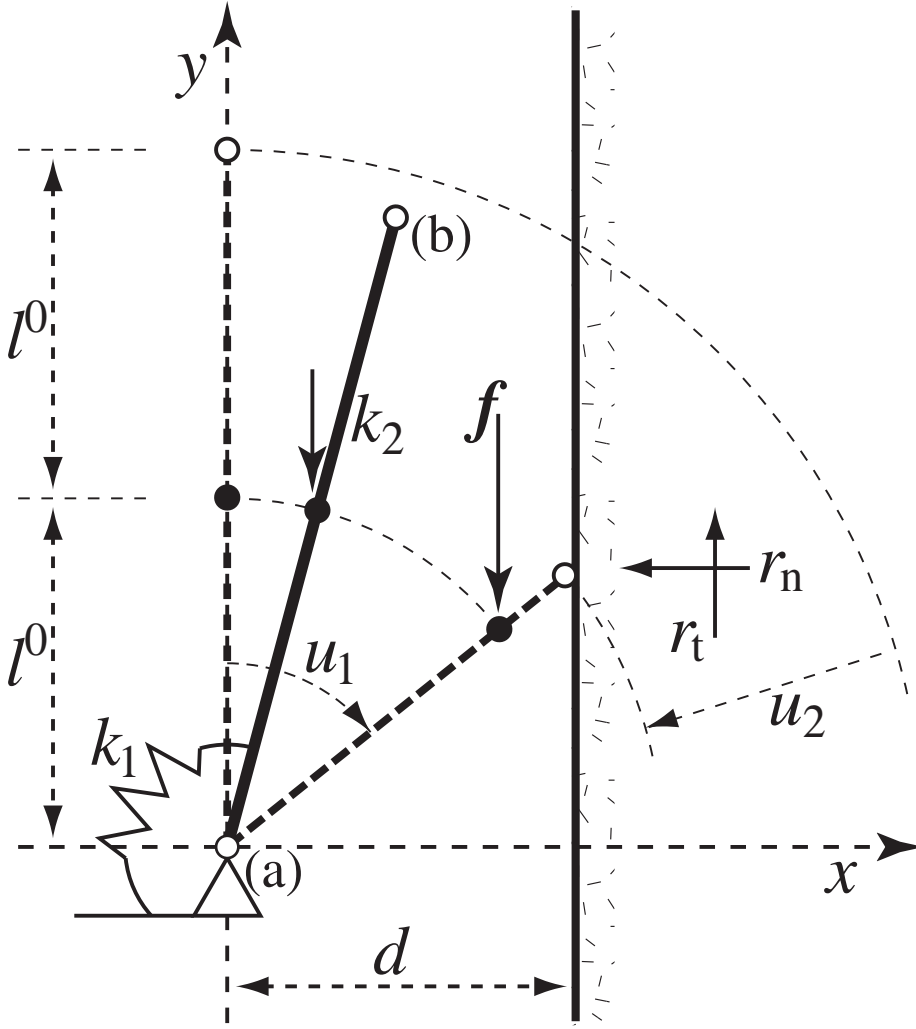
\Downarrow

$$(\mu r_n, r_t) \cdot (\lambda_n, \Delta u_t) \leq \epsilon, \quad \epsilon > 0$$

- solve the sequence of regularized problems
 - by $\epsilon \searrow 0$
 - by conventional SQP
 - Matlab Optimization Toolbox (Ver. 2.1)

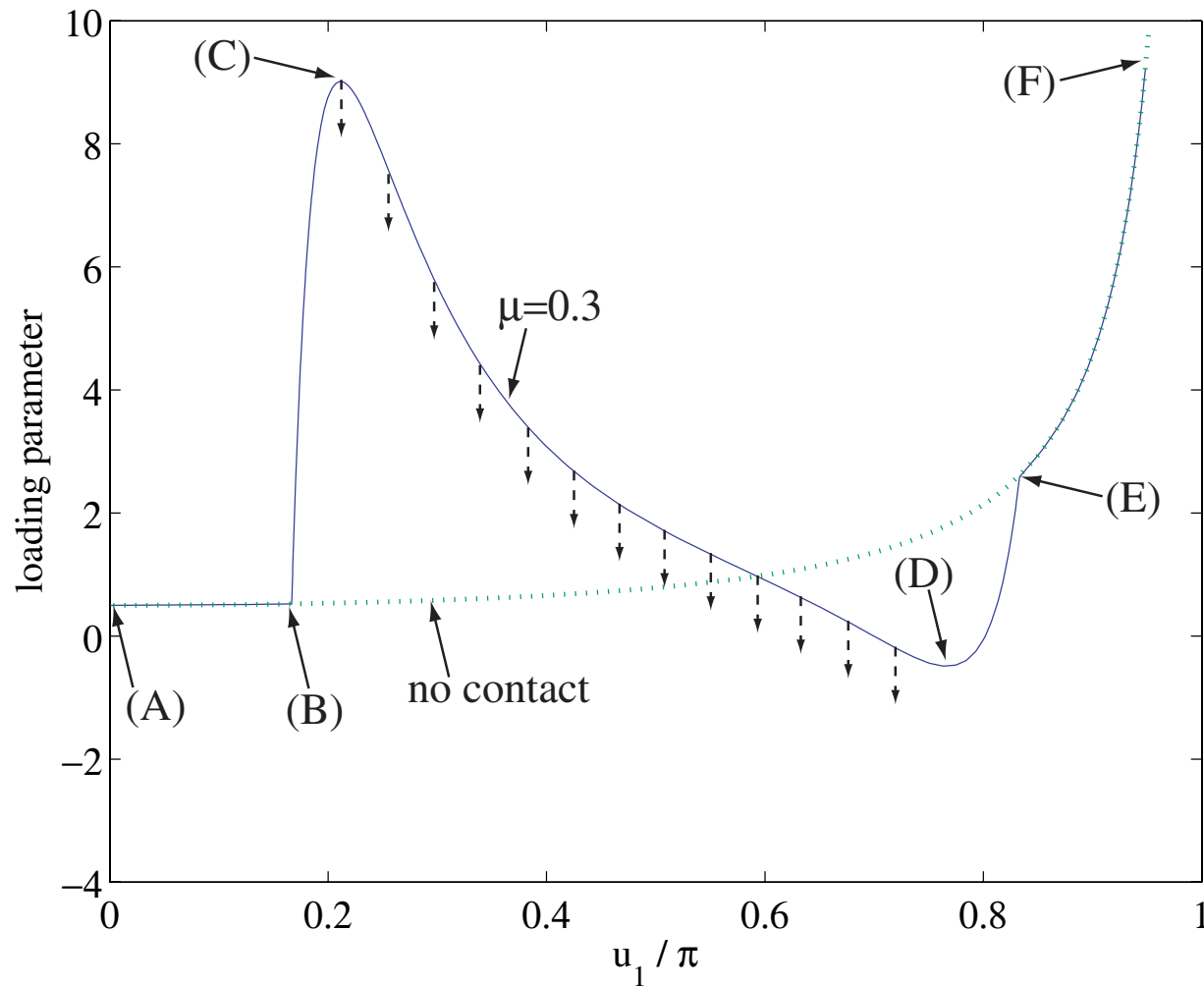
One-bar truss

- analogous to [Mróz 02]



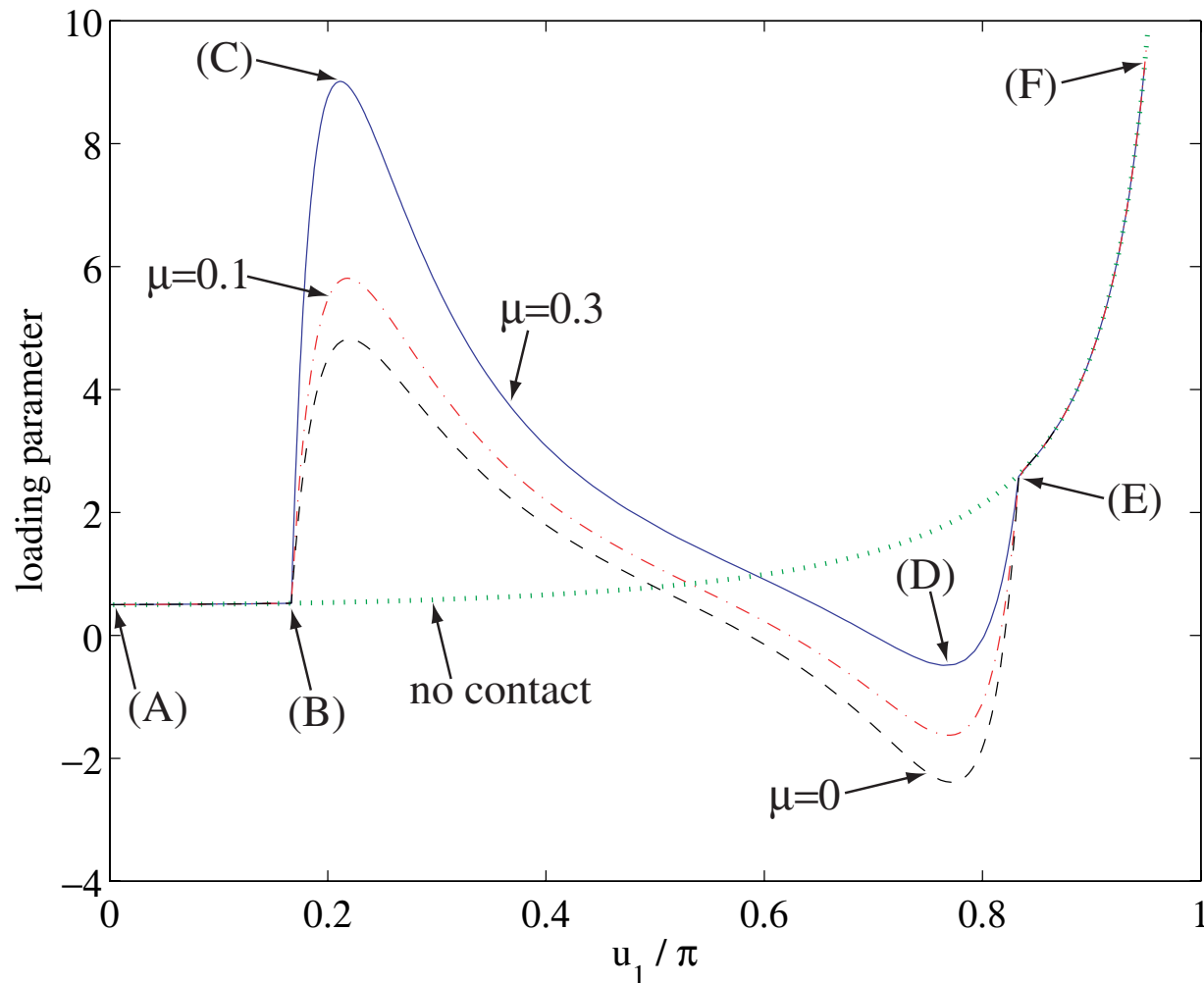
One-bar truss

- smooth limit points
- (C) \rightarrow (D) : bifurcation points

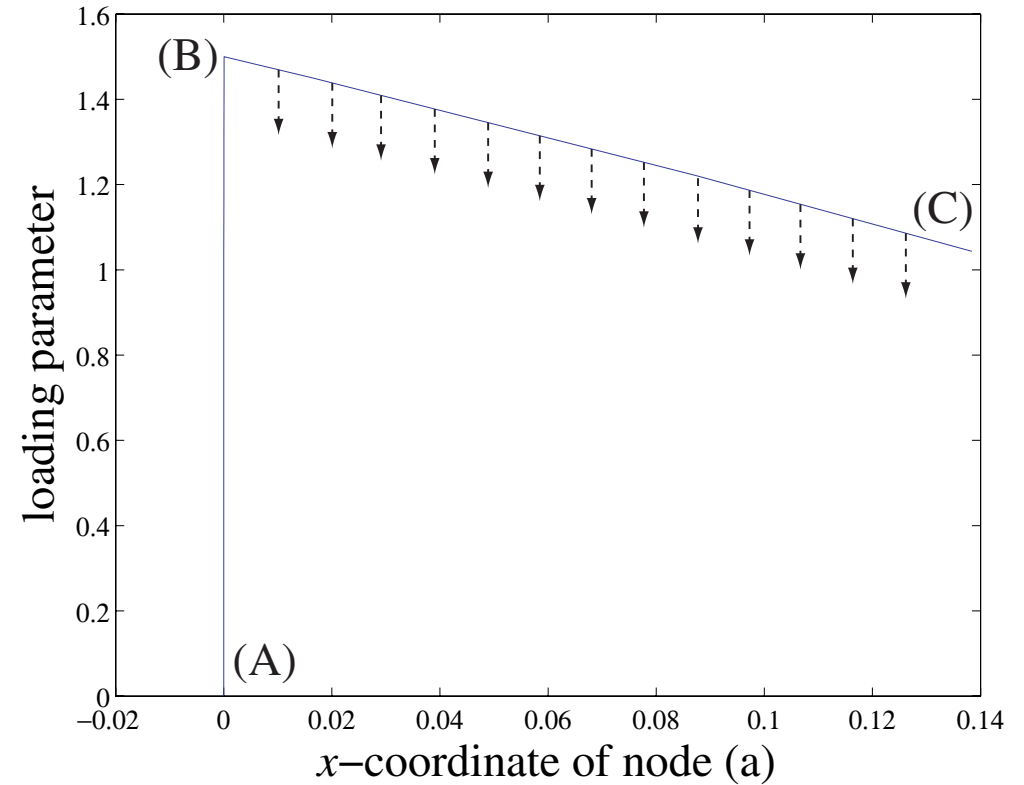
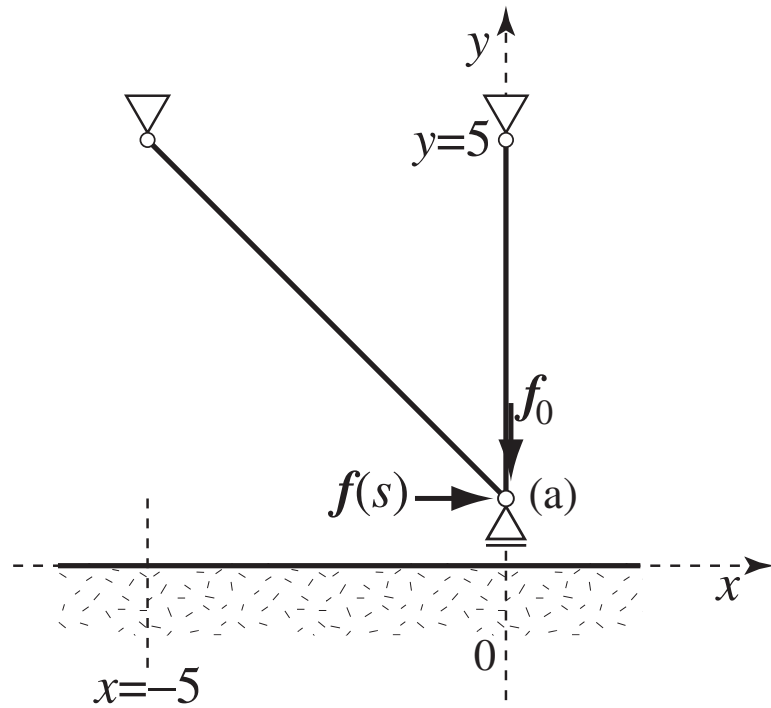


One-bar truss

- smooth limit points
- (C) → (D) : bifurcation points

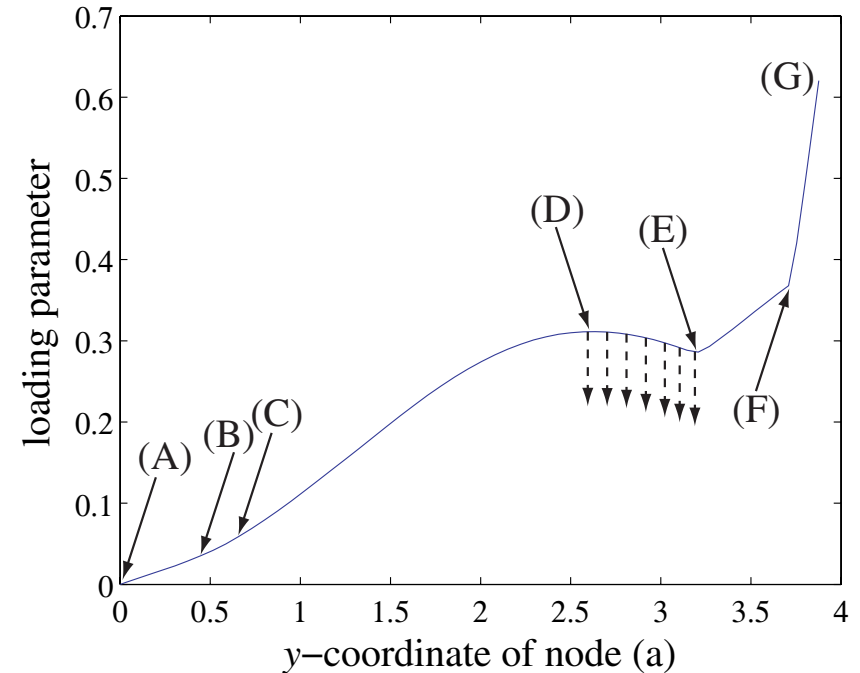
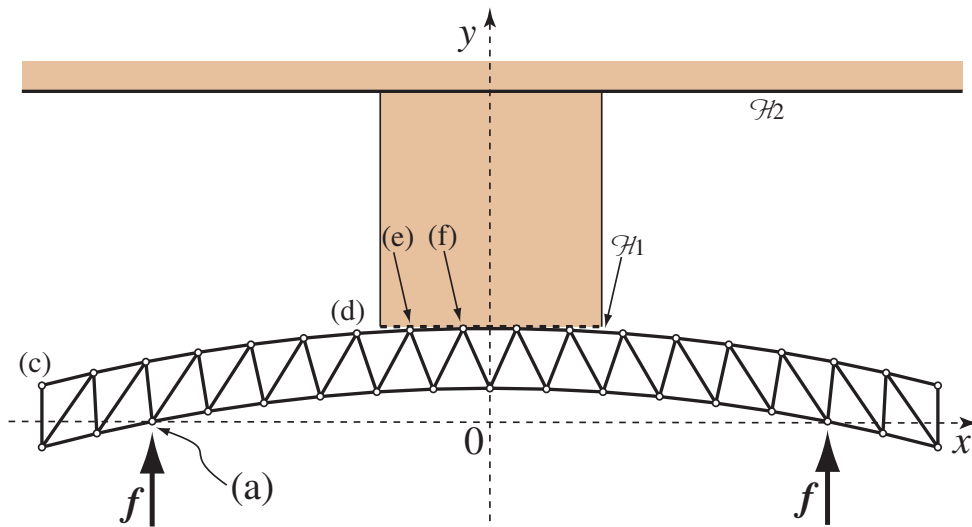


Ex. of [Klarbring 90]



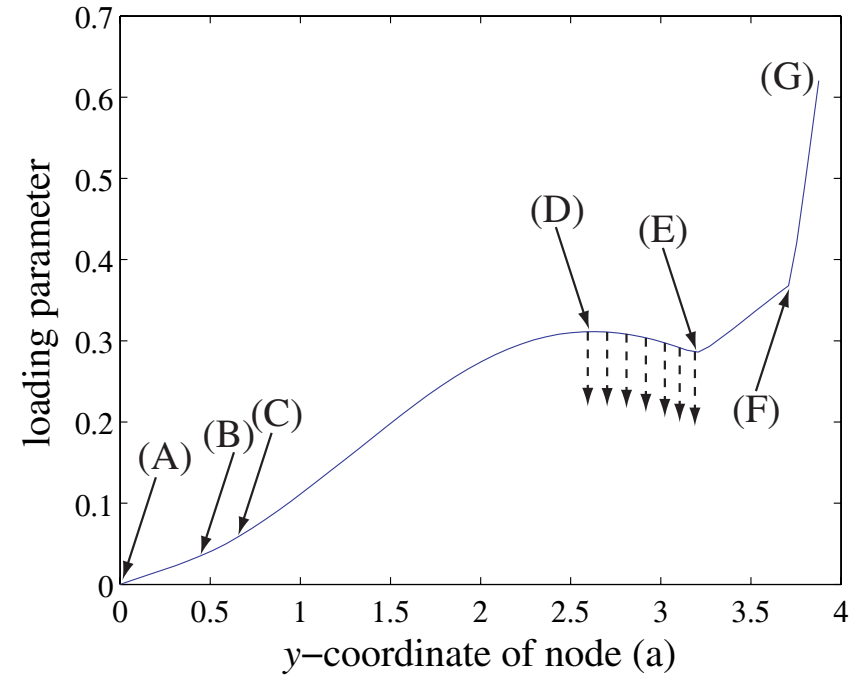
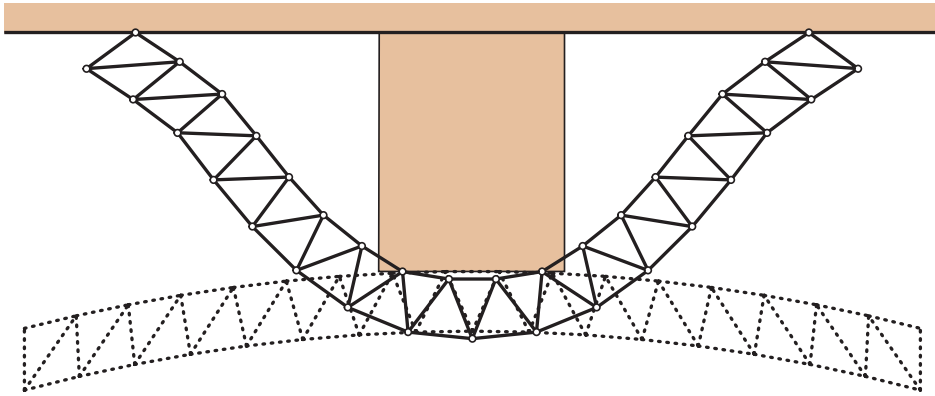
- $\mu = 1.5$
- (B) : nonsmooth limit point
- (B) \rightarrow (C) : bifurcation points

Arch-type truss



- (D) : smooth limit point
- (E) : nonsmooth limit point
- (D) → (E) : bifurcation points

Arch-type truss



- (D) : smooth limit point
- (E) : nonsmooth limit point
- (D) \rightarrow (E) : bifurcation points

Conclusion

- Arc-length method
 - with
 - contact, the Coulomb friction
 - choosing the path with the **max. dissipation**
 - geometrical nonlinearity
- MPEC is solved
 - at each increment
 - by using **regularization** and SQP method
- Trace equilibrium paths with
 - smooth / nonsmooth limit points
 - successive bifurcation points