

*Robust Truss Topology Optimization  
under Geometric Uncertainties  
via Semidefinite Programming*

Daiki Hashimoto, Yoshihiro Kanno

May 27, 2014

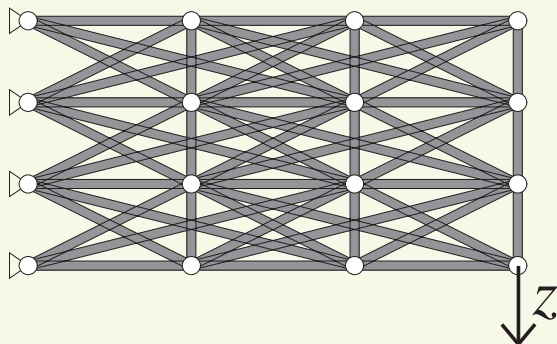
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# robust optimization

- nominal (i.e., conventional) compliance optimization

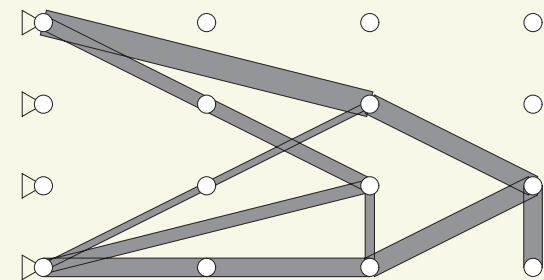
$$\begin{aligned} \min \quad & \pi(\mathbf{a}; \mathbf{z}) \\ \text{s. t.} \quad & \mathbf{a} \geq \mathbf{0}, \quad \text{vol}(\mathbf{a}) \leq \bar{V} \end{aligned}$$

- $\pi(\mathbf{a}; \mathbf{z})$  : compliance
- $\mathbf{a}$  : cross-sectional areas       $\mathbf{z}$  : data (e.g., external load)



ground struct.

→  
optimize



nominal opt. sol.

# robust optimization

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- $\pi(\mathbf{a}; \mathbf{z})$  : compliance
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- robust compliance optimization

$$\begin{aligned} \min \quad & \max\{\pi(\mathbf{a}; \mathbf{z}) \mid \mathbf{z} \in \mathcal{U}\} \\ \text{s. t.} \quad & \mathbf{a} \geq \mathbf{0}, \quad \text{vol}(\mathbf{a}) \leq \bar{V} \end{aligned}$$

- $\mathcal{U}$  : uncertainty set      (e.g., set of uncertain loads)
- obj. fcn.: worst value of compliance

# robust optimization

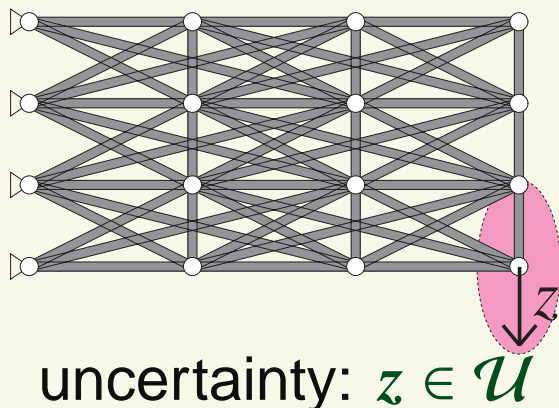
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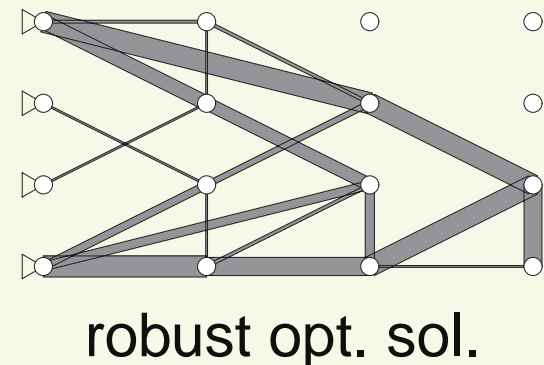
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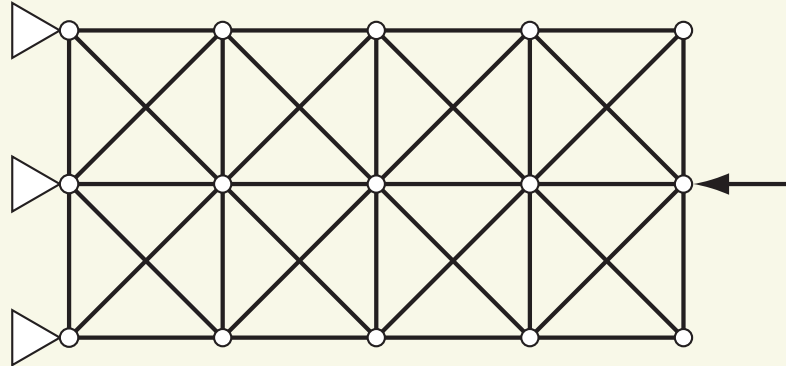


→  
optimize



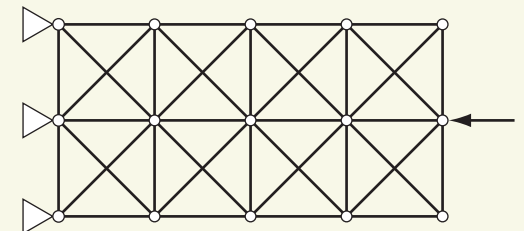
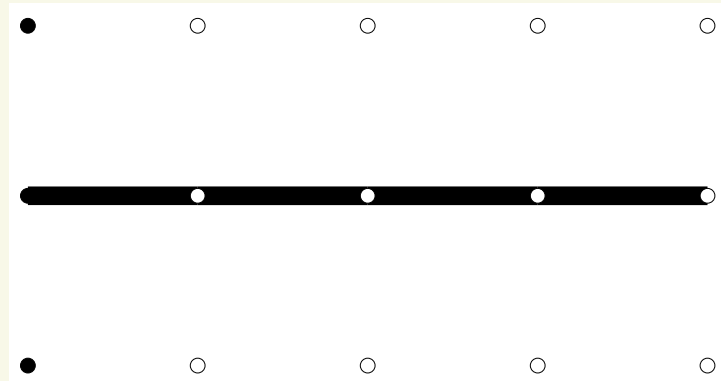
# geometric uncertainty: motivation

- compliance minimization of a truss
  - problem setting:



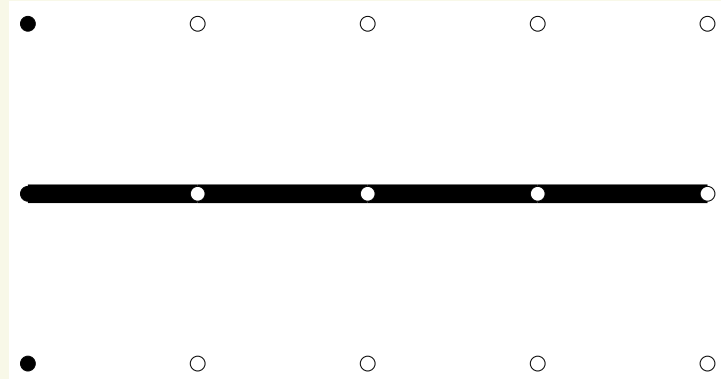
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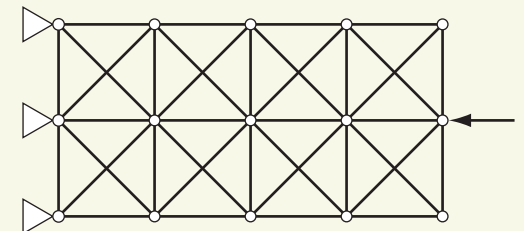


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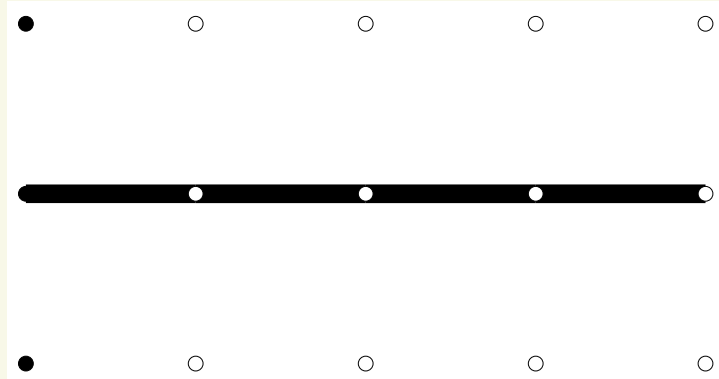


- unstable (kinematically indeterminate)
- unrealistic design under compression force
- imperfection in nodal locations  $\rightarrow$  no equilibrium state

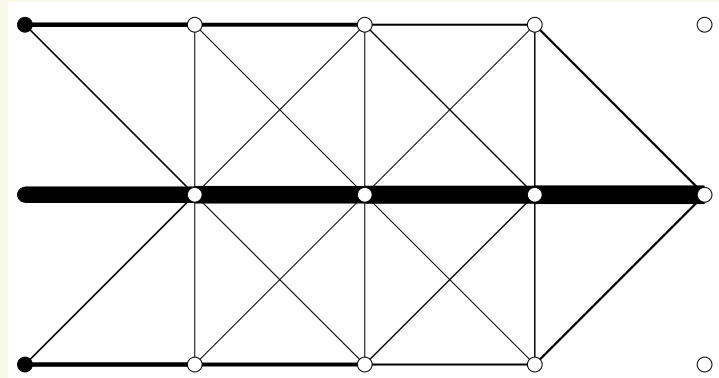


# geometric uncertainty: motivation

- compliance minimization of a truss
  - optimal solution:



- robust optimal solution:



- robustness against uncertainty in nodal locations



# robust truss optimization

- many studies concerning uncertain loads
  - possibilistic (i.e., non-probabilistic) model
    - given set of loads
    - min. the worst compliance [Ben-Tal & Nemirovski 97]  
[Cherkaev & Cherkaev 03] [Calafiore & Dabbene 08]  
[de Gournay, Allaire, & Jouve 08] [Takezawa, Nii, Kitamura, & Kogiso 11]

# robust truss optimization

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- uncertainty in nodal locations
  - probabilistic model [Guest & Igusa 08]
    - nodal locs.: random variables
    - min. the expected val. of compliance
  - possibilistic model [this study]

# uncertainty in boundary shape of continuum

- uniform manufacturing error
  - SIMP [Sigmund 09], [Wang, Lazarov, & Sigmund 11]
  - level-set method [Jang, van Dijk, van Keulen 12]
- probabilistic model
  - SIMP [Schevenels, Lazarov, & Sigmund 11] [Lazarov, Schevenels, & Sigmund 12]
  - level-set method [Chen & Chen 11]
- possibilistic model (1st order approx.) [Guo, Zhang, & Zhang 13]

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- conservative reformulation under large uncertainty
    - semidefinite programming (SDP) [this study]

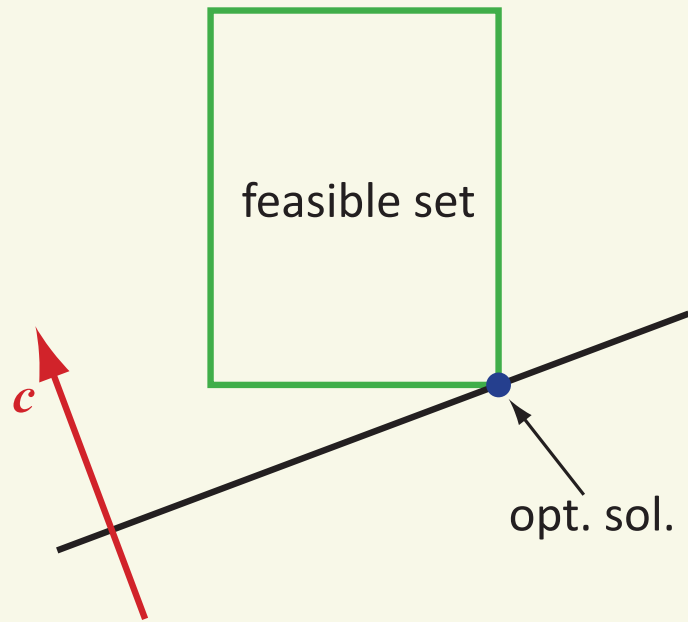
# SDP for robust optimization

- SDP — convex
  - compliance, load unc. [Ben-Tal & Nemirovski 97]
- nonlinear SDP — nonconvex
  - stress constraints, load unc. [Kanno & Takewaki 06]
  - stiffness unc. (safe approx.) [Guo, Bai, Zhang, & Gao 09]
  - stiffness & length unc. (safe approx.) [Guo, Du, & Gao 11]

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- SDP — convex
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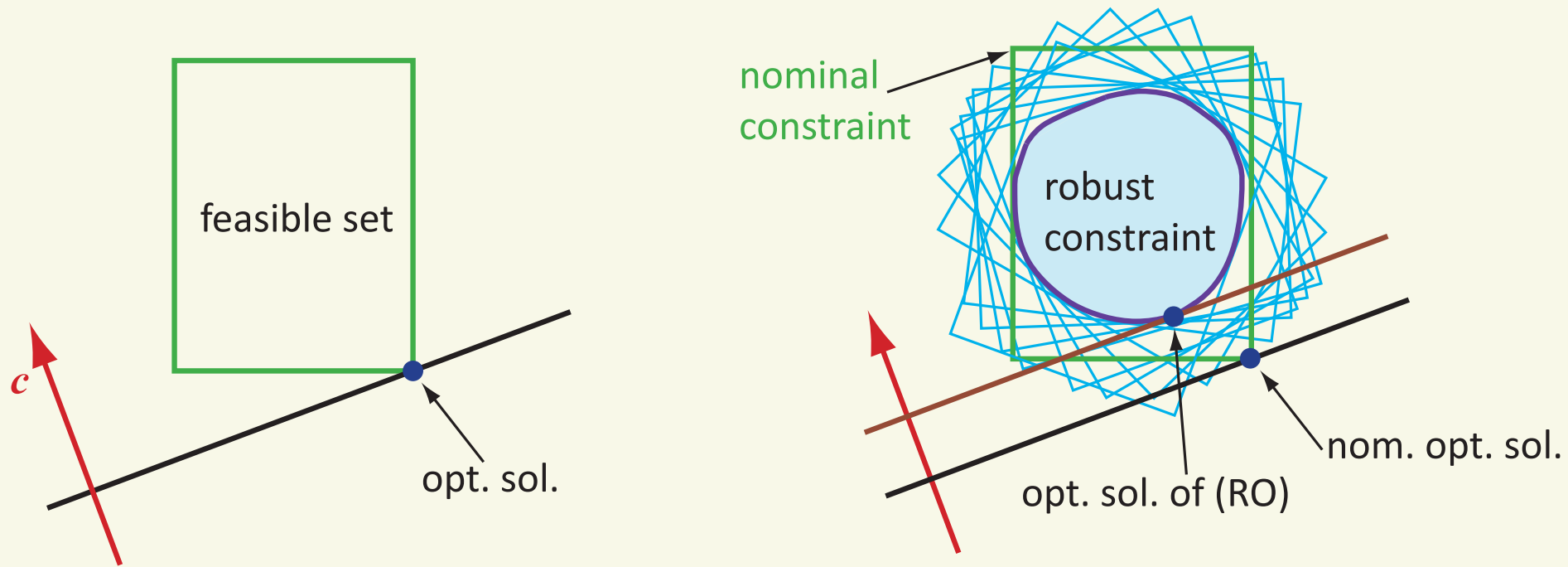
# robust opt. & conservative approx.



- nominal opt. prob.:

$$\begin{array}{ll} \min & \mathbf{c}^\top \mathbf{x} \\ \text{s. t.} & l_i \leq x_i \leq u_i \end{array}$$

# robust opt. & conservative approx.



- nominal opt. prob.:

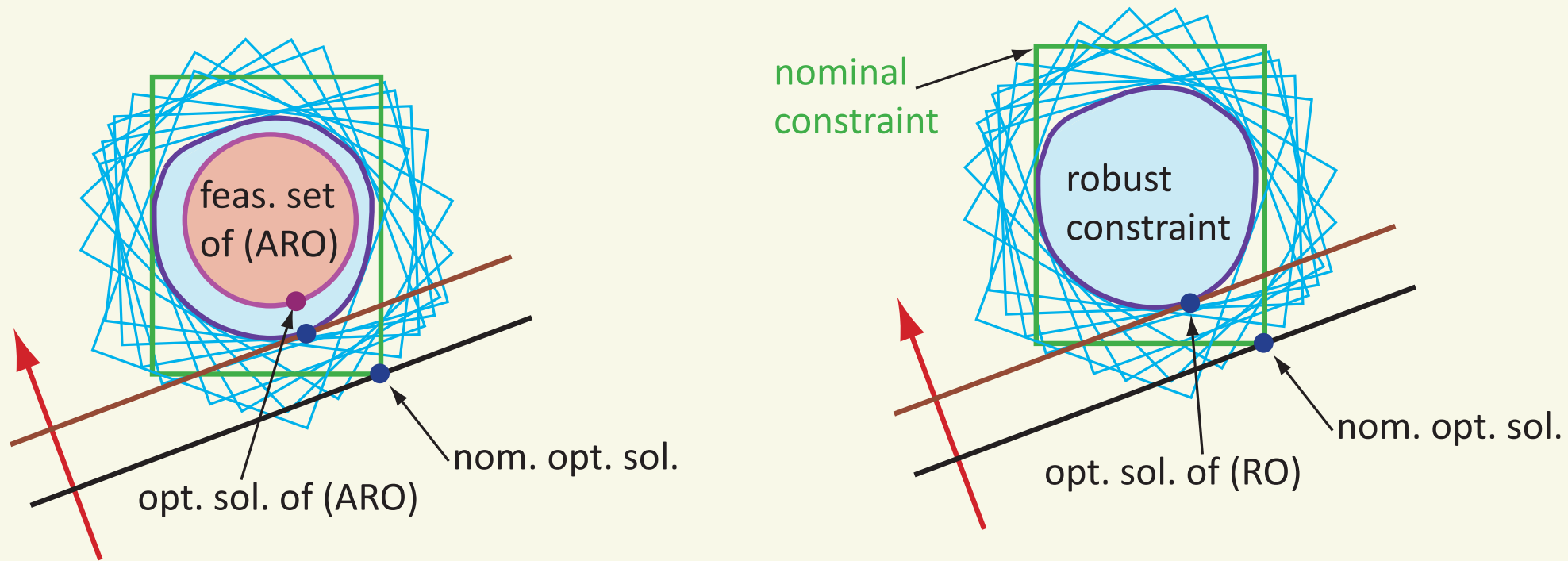
$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s. t.} \quad & l_i \leq x_i \leq u_i \end{aligned}$$

- robust opt. prob.:

$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s. t.} \quad & l_i(\mathbf{z}) \leq x_i \leq u_i(\mathbf{z}) \quad (\forall \mathbf{z} \in \mathcal{U}) \end{aligned} \quad \text{(RO)}$$



# robust opt. & conservative approx.



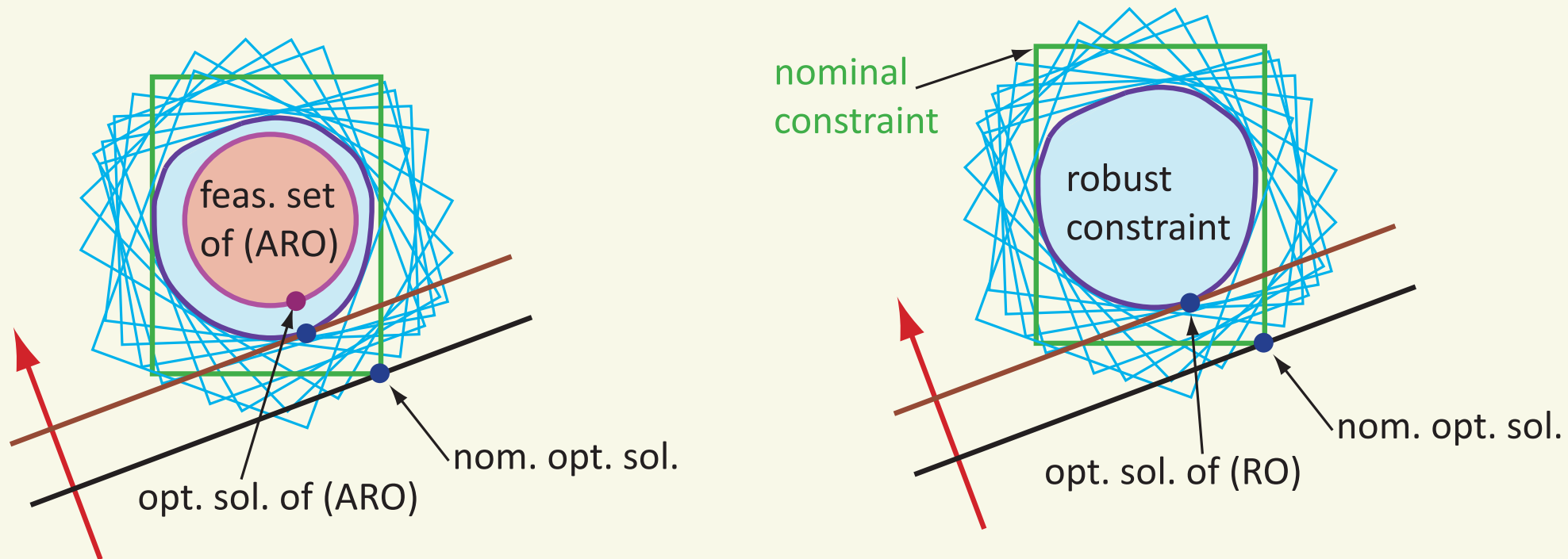
- conservative approx.:

$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s. t.} \quad & \mathbf{S}(\mathbf{x}) \geq \mathbf{O} \text{ (p.s.d.)} \end{aligned} \quad (\text{ARO})$$

- robust opt. prob.:

$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s. t.} \quad & l_i(\mathbf{z}) \leq x_i \leq u_i(\mathbf{z}) \quad (\forall \mathbf{z} \in \mathcal{U}) \end{aligned} \quad (\text{RO})$$

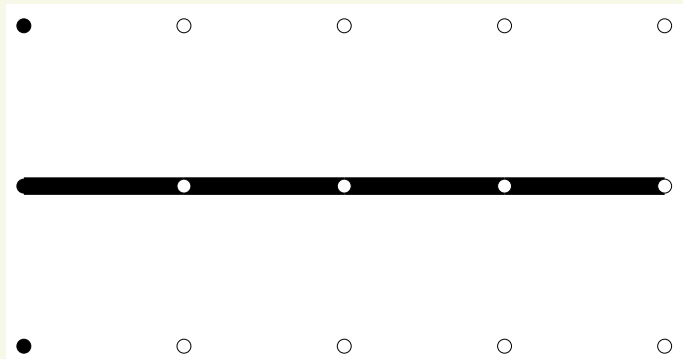
# robust opt. & conservative approx.



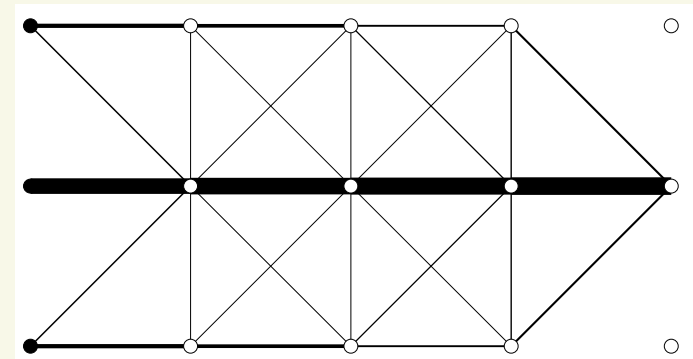
- feas. set of (ARO)  $\subseteq$  feas. set of (RO)
- opt. sol. of (ARO) satisfies robust cstr.
  - but may be “less optimal” than opt. sol. of (RO)
- (ARO) is easier than (RO)

# two properties of (ARO)

- stability
  - Under mild assumptions,
  - the opt. sol. of (ARO) is a stable truss.



nominal opt. sol.



opt. sol. of (ARO)

- zeroing
  - When  $r = 0$ ,
  - opt. sol. of (ARO) = nominal opt. sol.

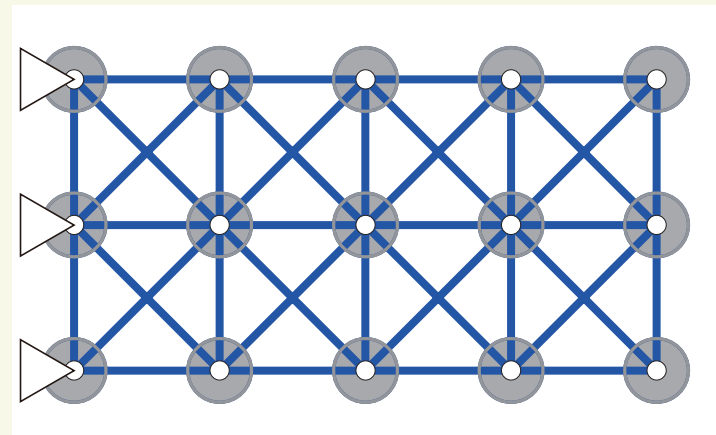
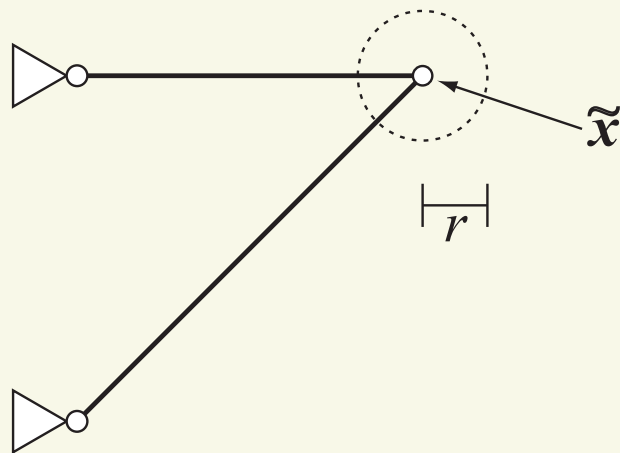
( $r$  : level of uncertainty)

# problem setting

- $\mathbf{x}$  : locations of nodes
- uncertainty set

$$\mathcal{U}_r = \{\tilde{\mathbf{x}} + \mathbf{A}\mathbf{z} \mid \|\mathbf{z}\| \leq r\}$$

- $\tilde{\mathbf{x}}$  : nominal locations
- $\mathbf{A}$  : constant matrix



Locations of all nodes can be uncertain.

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- $\tilde{\mathbf{x}}$  : nominal locations
- $\mathbf{A}$  : constant matrix
- nominal optimization

$$\begin{aligned} \min \quad & \text{compl}(\mathbf{a}; \tilde{\mathbf{x}}) \\ \text{s. t.} \quad & \mathbf{a} \geq \mathbf{0}, \quad \text{vol}(\mathbf{a}) \leq \bar{V} \end{aligned}$$

- $\mathbf{a}$  : cross-sectional areas — design variables

# reformulation

- nominal optimization

$$\begin{aligned} \min \quad & \text{compl}(\mathbf{a}; \tilde{\mathbf{x}}) \\ \text{s. t.} \quad & \mathbf{a} \geq \mathbf{0}, \quad \text{vol}(\mathbf{a}) \leq \bar{V} \end{aligned}$$

- SDP formulation

[Ben-Tal & Nemirovski 97]

$$\begin{aligned} \min \quad & w \\ \text{s. t.} \quad & \begin{bmatrix} w & \mathbf{f}^\top \\ \mathbf{f} & K(\mathbf{a}; \tilde{\mathbf{x}}) \end{bmatrix} \succeq O, \\ & \mathbf{a} \geq \mathbf{0}, \quad \text{vol}(\mathbf{a}) \leq \bar{V} \end{aligned}$$

$\mathbf{f}$  : load     $K$  : stiffness matrix

- robust optimization: robust SDP

$$\begin{aligned} \min \quad & w \\ \text{s. t.} \quad & \begin{bmatrix} w & \mathbf{f}^\top \\ \mathbf{f} & K(\mathbf{a}; \mathbf{x}) \end{bmatrix} \succeq O \quad (\forall \mathbf{x} \in \mathcal{U}_r) \quad \text{(RO)} \\ & \mathbf{a} \geq \mathbf{0}, \quad \text{vol}(\mathbf{a}) \leq \bar{V} \end{aligned}$$

# main result

- conservative approx. of (RO):

$$\begin{aligned}
 & \min \quad w \\
 & \text{s. t.} \quad \begin{bmatrix} & & \\ & w & \mathbf{f}^\top \\ & \mathbf{f} & \end{bmatrix} + \sum_{i \in \mathcal{E}} a_i \check{K}_i \begin{bmatrix} & & -rB_i^\top \\ & & \\ -rB_i & & \mathbf{b}_i \mathbf{b}_i^\top \end{bmatrix} \\
 & \quad \quad \quad + \begin{bmatrix} \text{diag}(\boldsymbol{\lambda}) & & \\ & & \\ & & -\sum_{i \in \mathcal{E}} \lambda_i C_i C_i^\top \end{bmatrix} \succeq \mathbf{O}, \\
 & \quad \quad \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad \mathbf{a} \geq \mathbf{0}, \quad \text{vol}(\mathbf{a}) \leq \bar{V}
 \end{aligned} \tag{ARO}$$

- SDP problem — convex
  - variables:  $\mathbf{a}, \boldsymbol{\lambda}, w$
  - can be solved with a primal-dual interior-point method

## derivation (1/2)

- robust constraint:

$$\left[ \begin{array}{c|c} w & \mathbf{f}^\top \\ \hline \mathbf{f} & K(\mathbf{a}; \mathbf{x}) \end{array} \right] \succeq \mathbf{O} \quad (\forall \mathbf{x} \in \mathcal{U}_r)$$

- derive a sufficient condition
- matrix on the left-hand side:

$$\Omega_0 + \sum_{i \in \mathcal{E}} a_i \Omega_i(\mathbf{x})$$

- with

$$\Omega_i(\mathbf{x}) = \kappa_i(\mathbf{x})(\hat{\mathbf{b}}_i + \hat{\mathbf{C}}_i \mathbf{x})(\hat{\mathbf{b}}_i + \hat{\mathbf{C}}_i \mathbf{x})^\top.$$



## derivation (2/2)

$$\forall \mathbf{x} \in \mathcal{U}_r : \quad \Omega_0 + \sum_{i \in \mathcal{E}} a_i \Omega_i(\mathbf{x}) \geq 0$$

$$\Leftrightarrow \forall \mathbf{x} \in \mathcal{U}_r, \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1} : \quad \boldsymbol{\xi}^\top \Omega_0 \boldsymbol{\xi} + \sum_{i \in \mathcal{E}} a_i \boldsymbol{\xi}^\top \Omega_i(\mathbf{x}) \boldsymbol{\xi} \geq 0$$

$$\Leftrightarrow \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1} : \quad \min_{\mathbf{x} \in \mathcal{U}_r} \left\{ \boldsymbol{\xi}^\top \Omega_0 \boldsymbol{\xi} + \sum_{i \in \mathcal{E}} a_i \boldsymbol{\xi}^\top \Omega_i(\mathbf{x}) \boldsymbol{\xi} \right\} \geq 0$$

$$\Leftarrow \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1} : \quad \boldsymbol{\xi}^\top \Omega_0 \boldsymbol{\xi} + \sum_{i \in \mathcal{E}} a_i \min_{\mathbf{x} \in \mathcal{U}_r} \left\{ \boldsymbol{\xi}^\top \Omega_i(\mathbf{x}) \boldsymbol{\xi} \right\} \geq 0$$

$$\Leftarrow \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1} : \quad \overbrace{\boldsymbol{\xi}^\top \Omega_0 \boldsymbol{\xi} + \sum_{i \in \mathcal{E}} a_i \check{\kappa}_i (\boldsymbol{\xi}^\top \hat{\mathbf{b}}_i \hat{\mathbf{b}}_i^\top \boldsymbol{\xi})}^p + 2r \min_{\eta_i \in \mathbb{R}} \{ \eta_i \hat{\mathbf{b}}_i^\top \boldsymbol{\xi} \mid |\eta_i| \leq \|\hat{\mathbf{C}}_i^\top \boldsymbol{\xi}\| \} \geq 0$$

$$\Leftrightarrow \forall \boldsymbol{\xi} \in \mathbb{R}^{p+1}, \forall \boldsymbol{\eta} \in \mathbb{R}^m : \quad \text{if } |\eta_i| \leq \|\hat{\mathbf{C}}_i^\top \boldsymbol{\xi}\| \text{ then}$$

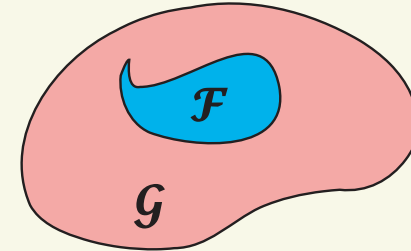
$$p - 2r a_i \check{\kappa}_i \sum_{i \in \mathcal{E}} \eta_i \hat{\mathbf{b}}_i^\top \boldsymbol{\xi} \geq 0.$$

## a mathematical tool

- ...for dealing with robust constraints
- $\mathcal{S}$ -lemma    [\[Pólik & Terlaky 07\]: survey](#)

# a mathematical tool

- ...for dealing with robust constraints
- $\mathcal{S}$ -lemma [Pólik & Terlaky 07]: survey
- $f_i(\mathbf{x}), g(\mathbf{x})$  : quadratic funcs.



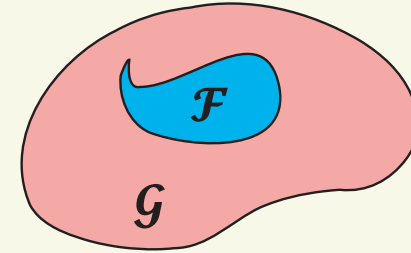
$$(a) f_1(\mathbf{x}) \geq 0 \quad \Rightarrow \quad g(\mathbf{x}) \geq 0$$



$$(b) \exists \lambda_1 \geq 0 : \quad g(\mathbf{x}) \geq \lambda_1 f_1(\mathbf{x}) \quad (\forall \mathbf{x})$$

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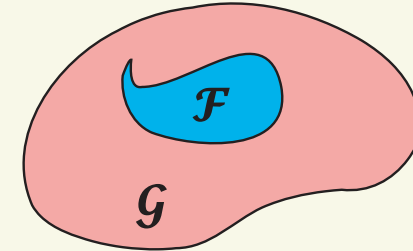


$$(b) \exists \lambda_1 \geq 0 : \quad g(\mathbf{x}) \geq \lambda_1 f_1(\mathbf{x}) \quad (\forall \mathbf{x})$$

- Farkas' lemma
  - $f_i(\mathbf{x}), g(\mathbf{x})$  : linear funcs.
  - (a') : “ $f_1(\mathbf{x}) \geq 0, g(\mathbf{x}) < 0$ ” has no solution.

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  - $f_i(\mathbf{x}), g(\mathbf{x})$  : quadratic funcs.



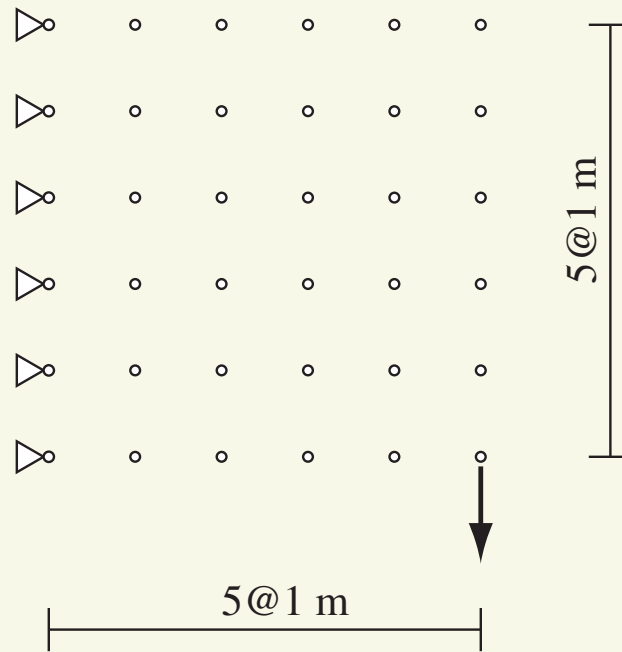
$$(a) \underbrace{f_1(\mathbf{x}) \geq 0, \dots, f_m(\mathbf{x}) \geq 0}_{\mathcal{F}} \Rightarrow \underbrace{g(\mathbf{x}) \geq 0}_{\mathcal{G}}$$

↑↑

$$(b) \exists \lambda_i \geq 0 : g(\mathbf{x}) \geq \lambda_1 f_1(\mathbf{x}) + \dots + \lambda_m f_m(\mathbf{x}) (\forall \mathbf{x})$$

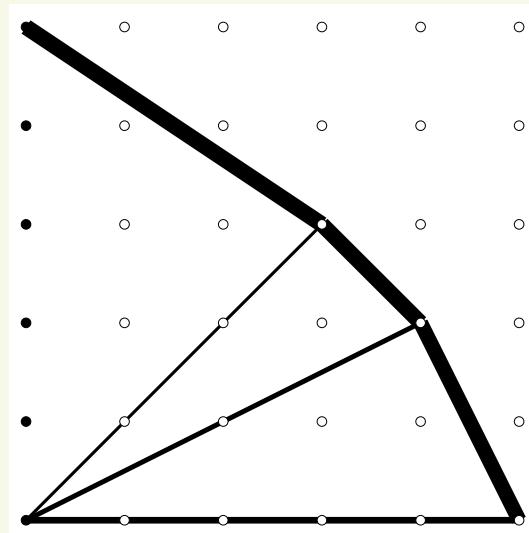
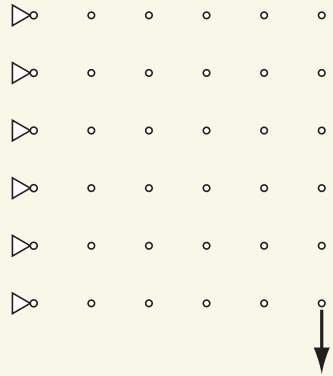
- (a) :  $\mathcal{F} \subseteq \mathcal{G}$   
( $\mathcal{F}$  : variation of compliance;  $\mathcal{G}$  : feasible set)
- (b) : p.s.d. constraint on a matrix  
→ constraint of SDP

## ex.) 418-bar truss



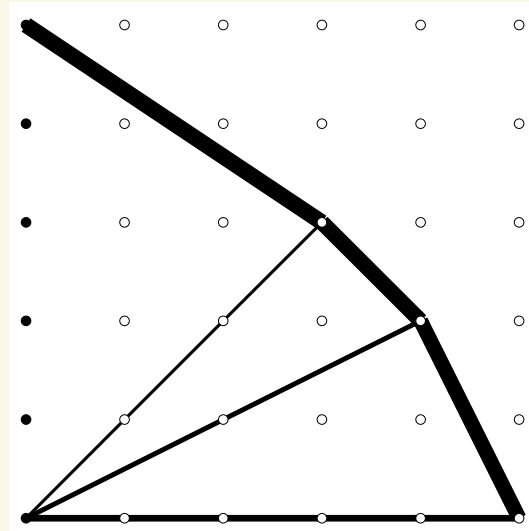
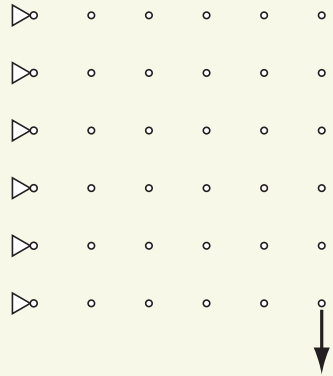
- ground structure
  - Any two nodes are connected by a member.
  - Overlapping members are removed.
- uncertainty: locations of all nodes
  - incl. supports

# ex.) 418-bar truss

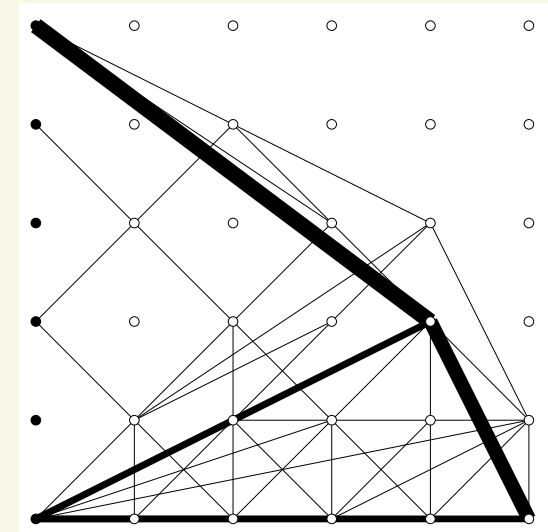


nominal opt. sol.

# ex.) 418-bar truss



nominal opt. sol.

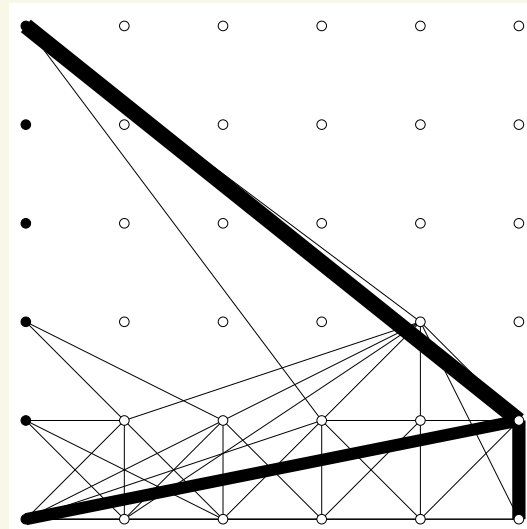


$r = 0.02 \text{ m}$

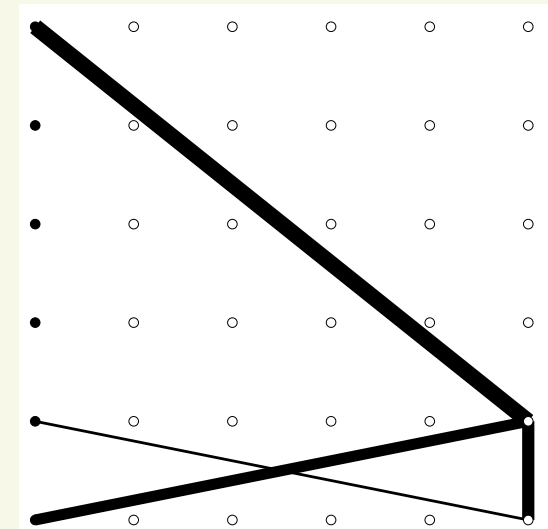
very large  $r$



few nodes



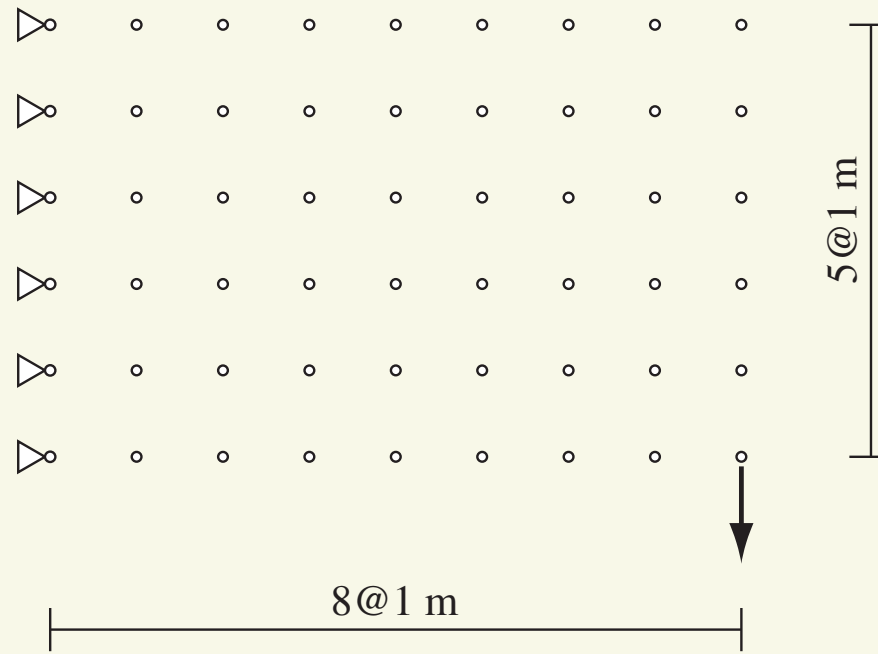
$r = 0.05 \text{ m}$



$r = 0.1 \text{ m}$

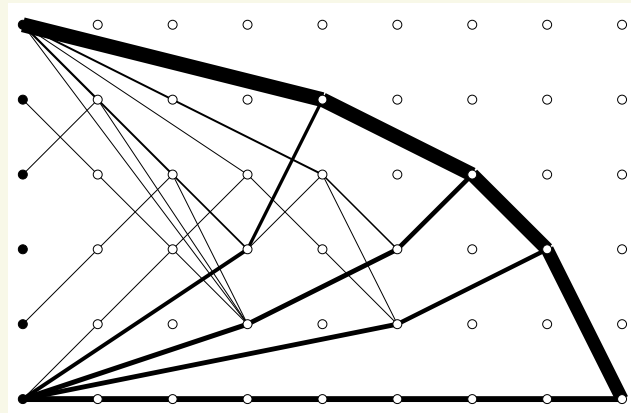
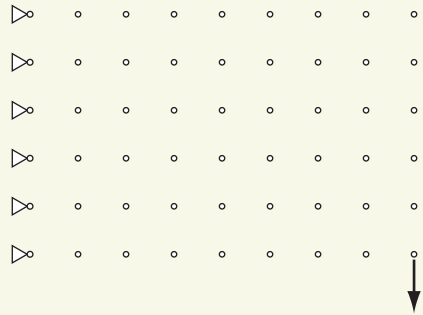


## ex.) 919-bar truss



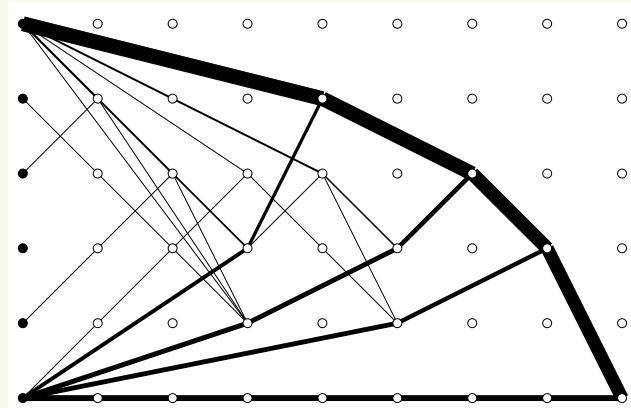
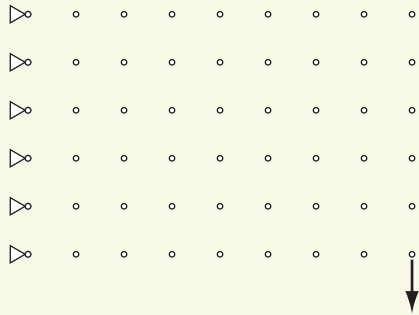
- uncertainty: locations of all nodes
  - incl. supports

# ex.) 919-bar truss

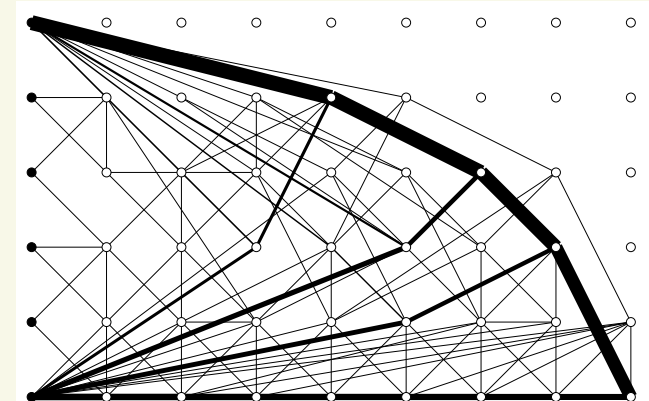


nominal opt.

# ex.) 919-bar truss



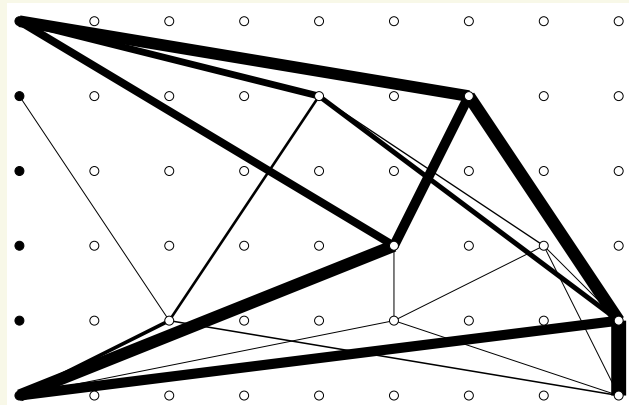
nominal opt.



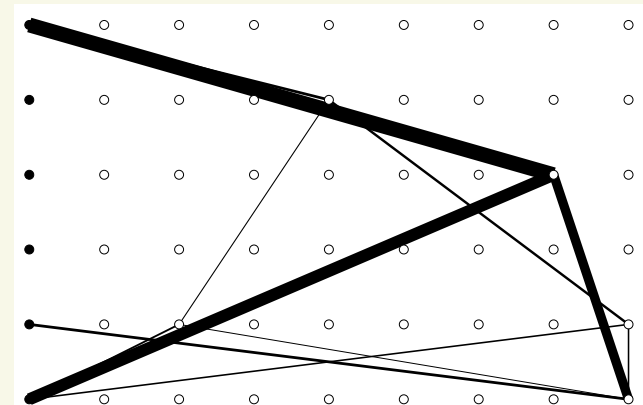
$r = 0.02$  m

$r$ (m)	Time (s)
0	10.2
0.02	1285.2
0.05	863.2
0.10	685.8

(SeDuMi 1.3)



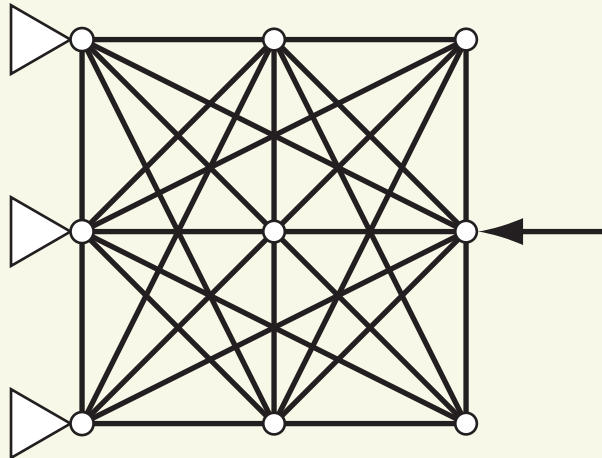
$r = 0.05$  m



$r = 0.1$  m

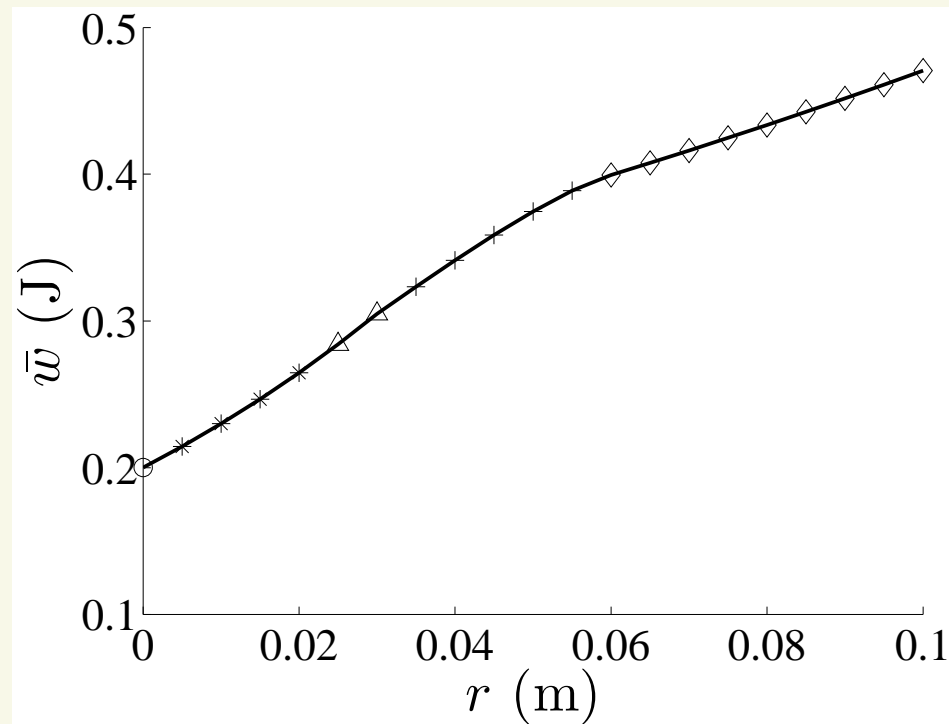
## ex.) effect of uncertainty magnitude

- level of uncertainty:  $r$ 
  - 0 (= no uncertainty)  $\leftrightarrow$  0.1 m ( $\approx$  10% uncertainty)
- opt. val. of SDP:  $\bar{w}$  ( $\approx$  worst compliance)



## ex.) effect of uncertainty magnitude

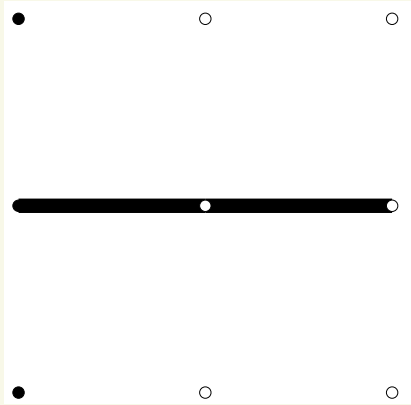
- level of uncertainty:  $r$ 
  - 0 (= no uncertainty)  $\leftrightarrow$  0.1 m ( $\approx$  10% uncertainty)
- opt. val. of SDP:  $\bar{w}$  ( $\approx$  worst compliance)



- robustness curve (trade off)
  - performance requirement  $\uparrow$  (i.e.,  $\bar{w} \downarrow$ )  $\leftrightarrow$  robustness  $r \downarrow$

## ex.) effect of uncertainty magnitude

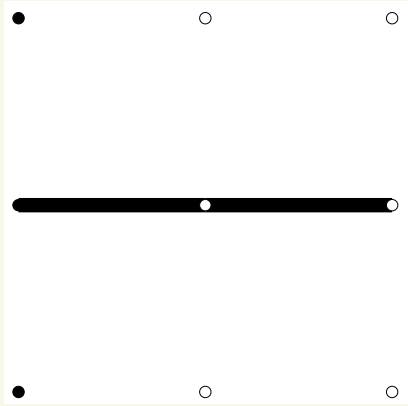
- level of uncertainty:  $r$  vs. variation of topology



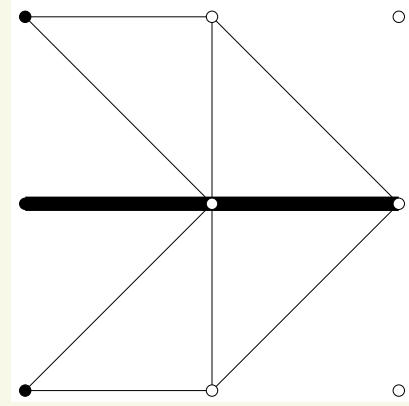
$$r = 0$$

# ex.) effect of uncertainty magnitude

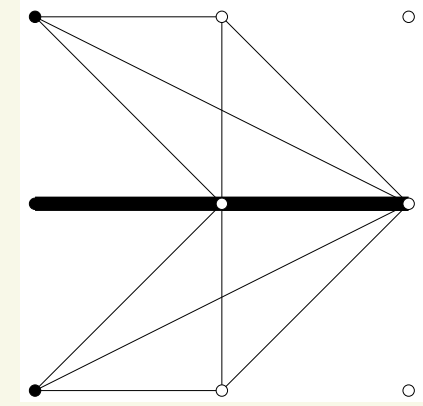
- level of uncertainty:  $r$  vs. variation of topology



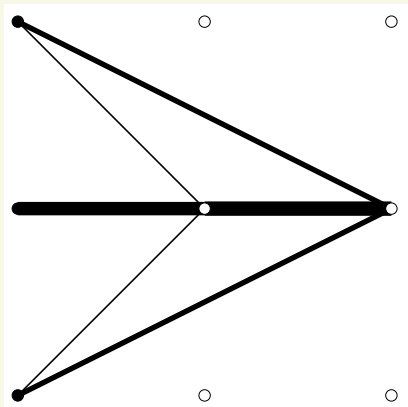
$r = 0$



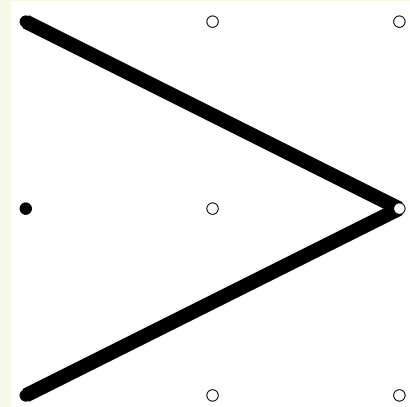
$0.05 \leq r \leq 0.02$



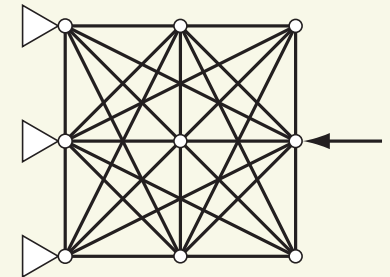
$0.025 \leq r \leq 0.03$



$0.035 \leq r \leq 0.055$



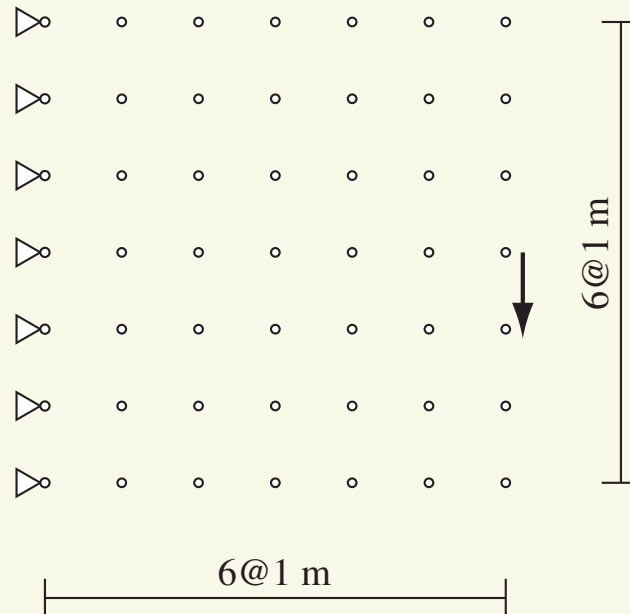
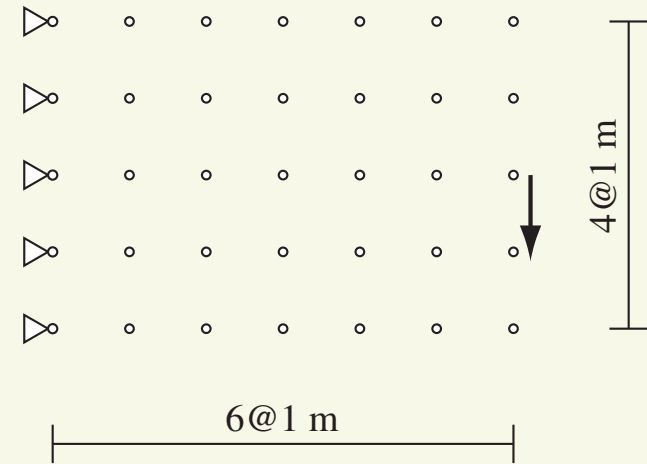
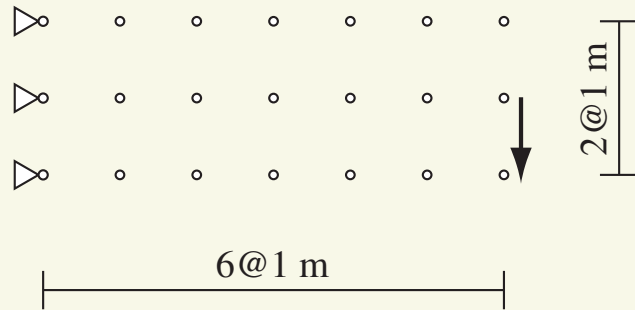
$0.06 \leq r \leq 0.1$



- large  $r \leftrightarrow$  fewer nodes

# ex.) more examples

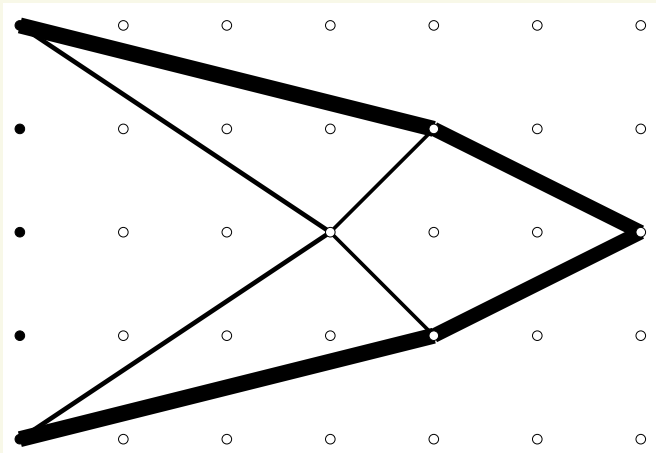
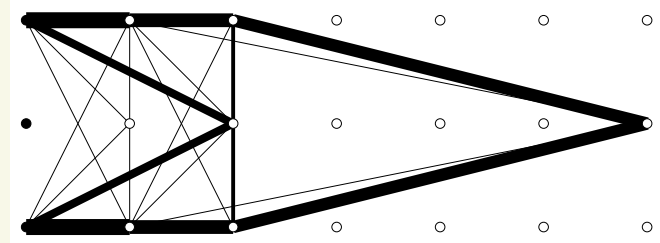
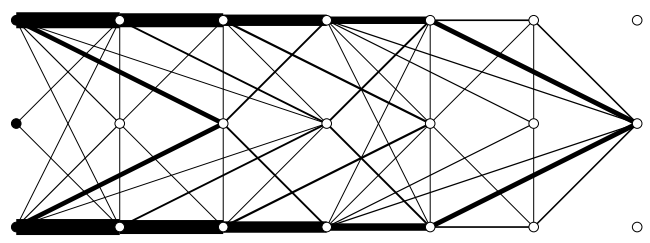
- ground structures



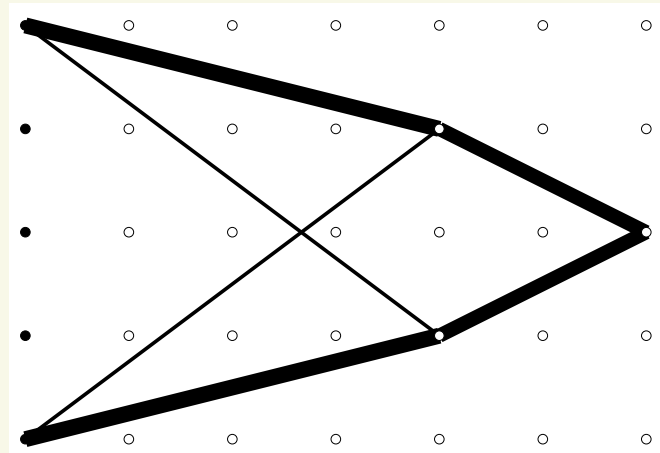


## ex.) more examples

- solutions



nominal opt.

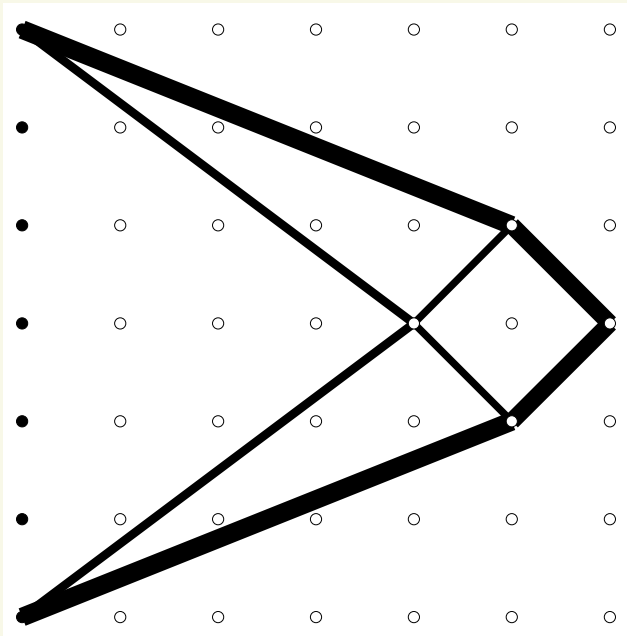


robust opt. ( $r = 0.05$  m)

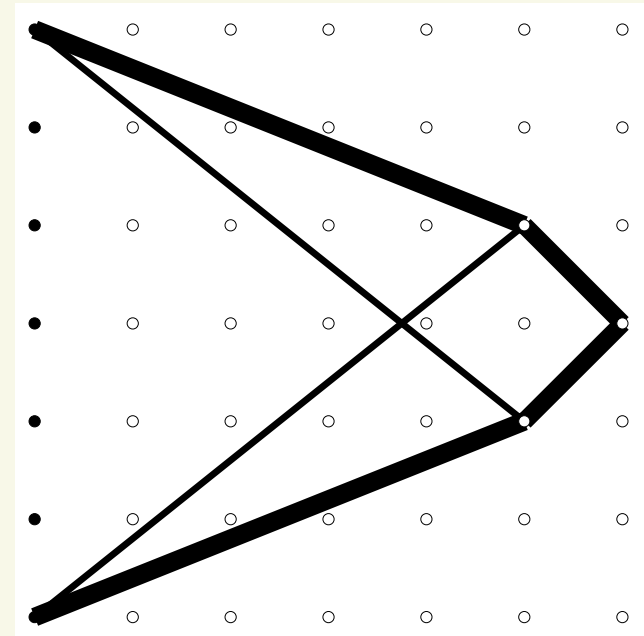
- Nominal opt. sol. is also stable.
- Robust opt. sol. has less nodes.

## ex.) more examples

- solutions



nominal opt.



robust opt. ( $r = 0.05$  m)

- Nominal opt. sol. is also stable.
- Robust opt. sol. has less nodes.

# conclusions

- robust truss optimization
  - uncertainty: locations of nodes  $\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{A}\mathbf{z}$  ( $\|\mathbf{z}\| \leq r$ )
  - worst-case compliance  $\rightarrow$  Min.
- SDP (semidefinite programming) approach:
  - conservative approximation
    - (SDP)  $\subseteq$  (RO)
    - providing an upper bound
  - convex optimization
    - primal-dual interior-point method
  - stability of the optimal solution
  - zeroing